



Discrete approximation of the viscous HJ equation

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Abstract

We consider a stochastic discretization of the stationary viscous Hamilton–Jacobi equation on the flat d –dimensional torus \mathbb{T}^d associated with a Hamiltonian, convex and superlinear in the momentum variable. We show that each discrete problem admits a unique continuous solution on \mathbb{T}^d , up to additive constants. By additionally assuming a technical condition on the associated Lagrangian, we show that each solution of the viscous Hamilton–Jacobi equation is the limit of solutions of the discrete problems, as the discretization step goes to zero.

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Introduction

Several authors have considered the approximation of the value function in continuous time Optimal Control by means of the value function given by a discrete time Dynamical Programming Principle. The convergence of this approximation is in fact the basis for computational methods of solution of the corresponding Hamilton–Jacobi–Bellman equation. We can mention the book [1] and the articles [3–6,9,13] where convergence is proved on different settings.

In this paper, we propose a stochastic version of this discretization so to approximate the solutions of a viscous Hamilton–Jacobi equation of the kind

$$-\Delta u + H(x, Du) = \alpha_0 \quad \text{in } \mathbb{T}^d, \quad (1)$$

where \mathbb{T}^d is the flat d -dimensional torus and the Hamiltonian $H : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a continuous function, convex and superlinear in the momentum variable. Under suitable assumptions on H , there is a unique real constant α_0 such that equation (1) admits solutions in the viscosity sense. This constant α_0 is often termed *ergodic constant* or *Mañé critical value*. Furthermore, solutions to (1) are unique, up to additive constants, and are of class C^2 , hence they solve the equation (1) in the classical sense.

Solutions to (1) can be also regarded as fixed points, for every $t > 0$, of the operator $\mathcal{S}(t) : C(\mathbb{T}^d) \rightarrow C(\mathbb{T}^d)$, defined on the space $C(\mathbb{T}^d)$ of continuous \mathbb{Z}^d -periodic function on \mathbb{R}^d , as follows:

$$(\mathcal{S}(t)u)(x) = \inf_v \mathbb{E} \left[u(Y_x(t)) + \int_0^t (L(Y_x(s), -v(s)) + \alpha_0) ds \right] \quad (2)$$

for every $x \in \mathbb{T}^d$ and $t > 0$. Here L is the Lagrangian associated to H via the Legendre–Fenchel transform, $v : [0, \infty) \times \Omega \rightarrow \mathbb{R}^d$ is a control process satisfying suitable measurability conditions and Y_x is the solution of the following Stochastic Differential Equation

$$\begin{cases} dY_x(t) = v(t) dt + \sqrt{2} dW_t \\ Y_x(0) = x, \end{cases} \quad (3)$$

where W_t denotes a standard Brownian motion on \mathbb{R}^d , defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In the formula (2), the symbol \mathbb{E} stands for the expectation with respect to the probability measure \mathbb{P} and the minimization is performed by letting v vary in a proper class of admissible control processes.

Given a solution u of (1), which is also a (stationary) solution of the evolution equation

$$\frac{\partial}{\partial t} v - \Delta v + H(x, D_x v) = \alpha_0 \quad \text{in } (0, +\infty) \times \mathbb{T}^d,$$

with initial condition $v(0, x) = u(x)$, another way of representing the function $v(t, x) = u(x)$ is to apply the approximation scheme based on the Trotter–Kato theo-

rem. For $t > 0$ we define the heat and Hopf-Lax operators $\mathcal{S}_t^H, \mathcal{S}_t^{HJ} : C(\mathbb{T}^d) \rightarrow C(\mathbb{T}^d)$ by

$$\mathcal{S}_t^H \varphi = \eta^t * \varphi,$$

where $\eta^t * \varphi$ is the convolution of the function φ with the heat kernel

$$\eta^t(y) := (4\pi t)^{-\frac{d}{2}} e^{-\frac{|y|^2}{4t}},$$

and

$$(\mathcal{S}_t^{HJ} \varphi)(x) = \inf_{\gamma} \left(\varphi(\gamma(t)) + \int_0^t L(\gamma(s), -\dot{\gamma}(s)) ds \right) + t\alpha_0,$$

where $\gamma : [0, t] \rightarrow \mathbb{T}^d$ ranges over all curves (in a general class) satisfying $\gamma(0) = x$. Note that the functions $v(t, x) = (\mathcal{S}_t^H \varphi)(x)$ and $w(t, x) = (\mathcal{S}_t^{HJ} \varphi)(x)$ should be the solutions of $(\partial/\partial t)v - \Delta v = 0$ and $(\partial/\partial t)w + H(x, D_x w) = \alpha_0$ in $(0, +\infty) \times \mathbb{T}^d$, respectively, with the initial condition $v(0, x) = w(0, x) = \varphi(x)$. In view of the Trotter-Kato theorem, under appropriate assumptions, one may expect the convergence

$$\lim_{n \rightarrow +\infty} \left(\mathcal{S}_{t/n}^H \circ \mathcal{S}_{t/n}^{HJ} \right)^n u = \lim_{n \rightarrow +\infty} \left(\mathcal{S}_{t/n}^{HJ} \circ \mathcal{S}_{t/n}^H \right)^n u = u \quad \text{in } C(\mathbb{T}^d). \tag{4}$$

It is unlikely, but if we could find a common fixed point $u_0 \in C(\mathbb{T}^d)$ of the family of operators $\mathcal{S}_t^{HJ} \circ \mathcal{S}_t^H$ with small $t > 0$, then we would have $(\mathcal{S}_{t/n}^{HJ} \circ \mathcal{S}_{t/n}^H)^n u_0 = u_0$, which would imply by the Trotter-Kato theorem that u_0 is a solution of (1). Notice here that α_0 is assumed to be given although the constant α_0 is also an unknown in problem (1).

Motivated by these interpretations of the viscous Hamilton–Jacobi equation, for every fixed $\tau > 0$, we introduce an operator $\mathcal{L}_\tau : C(\mathbb{T}^d) \rightarrow C(\mathbb{T}^d)$ defined as follows:

$$\mathcal{L}_\tau u(x) := \min_{q \in \mathbb{R}^d} \left(\tau L(x, -q) + (\eta^\tau * u)(x + \tau q) \right) \quad \text{for every } x \in \mathbb{R}^d.$$

This discretized operator \mathcal{L}_τ is an obvious approximation of the operator $\mathcal{S}_\tau^{HJ} \circ \mathcal{S}_\tau^H$, with the term $\tau\alpha_0$ missing, which is to be found in the next theorem.

Theorem 1 *Let $L : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfy conditions (L1)–(L2) below. Then there exists a unique constant $\alpha_\tau \in \mathbb{R}$ for which the equation*

$$\mathcal{L}_\tau u = u - \tau\alpha_\tau \quad \text{in } \mathbb{T}^d \tag{5}$$

admits a solution $u \in C(\mathbb{T}^d)$. Furthermore, solutions are unique, up to additive constants.

Our main result is the following.

Theorem 2 Assume that $L : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfy conditions **(L1)**-**(L2)** below, together with

$$L(x + h, q + k) + L(x - h, q - k) - 2L(x, q) \leq M_\gamma(|h|^\gamma + |k|) \tag{L3}$$

for all $(x, q) \in \mathbb{T}^d \times \mathbb{R}^d$ and $h, k \in \overline{B}_{R_d}$ with $R_d := \sqrt{d}/2$, for some constants $\gamma \in (0, 1)$ and $M_\gamma > 0$. Let $x_0 \in \mathbb{T}^d$ be fixed and denote by u_τ the unique solution of

$$\mathcal{L}_\tau u = u - \tau \alpha_\tau \quad \text{in } \mathbb{T}^d$$

such that $u_\tau(x_0) = 0$. Then the family $\{u_\tau \mid \tau \in (0, 1)\}$ is equi-bounded and equi-continuous in $C(\mathbb{T}^d)$ and (α_τ, u_τ) converges to (α_0, u) in $\mathbb{R} \times C(\mathbb{T}^d)$, as $\tau \rightarrow 0$, where u is the unique viscosity solution to

$$-\Delta u + H(x, Du) = \alpha_0 \quad \text{in } \mathbb{T}^d, \tag{6}$$

satisfying $u(x_0) = 0$.

The paper is organized as follows: Section 1 contains the standing assumptions and some preliminary facts on the viscous Hamilton–Jacobi equation (1). In Sect. 2 we introduce the discrete operator \mathcal{L}_τ and study its main properties, in particular we prove Theorem 1. In Sect. 3 we prove equi-continuity of the solutions of the discrete problems. Section 4.1 is devoted to the proof Theorem 2, while Section 4.2 contains some examples for which the assertion of Theorem 2 holds true.

1 The viscous Hamilton–Jacobi equation

Throughout the paper, we will call Lagrangian a continuous function $L : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, which is \mathbb{Z}^d -periodic in the space variable x . Equivalently, L can be thought as defined on the tangent bundle $\mathbb{T}^d \times \mathbb{R}^d$ of the flat d -dimensional torus \mathbb{T}^d . We will assume L to satisfy the following hypotheses:

(L1) (Convexity) for every $x \in \mathbb{R}^d$, the map $q \mapsto L(x, q)$ is convex on \mathbb{R}^d .

(L2) (Superlinearity) $\inf_{x \in \mathbb{R}^d} \frac{L(x, q)}{|q|} \rightarrow +\infty$ as $|q| \rightarrow +\infty$.

To any such Lagrangian, we can associate a Hamiltonian function $H : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ via the Legendre-Fenchel transform:

$$H(x, p) := \sup_{q \in \mathbb{R}^d} \{ \langle p, q \rangle - L(x, q) \}. \tag{1.7}$$

Such a function H is clearly \mathbb{Z}^d -periodic in x . Furthermore, it satisfies convexity and superlinearity conditions analogous to **(L1)** and **(L2)**, to which we shall refer as **(H1)**

and **(H2)** in the sequel. Later in the paper, we will assume L to satisfy the following additional assumption:

(L3) there exist constants $\gamma \in (0, 1)$ and $M_\gamma > 0$ such that

$$L(x + h, q + k) + L(x - h, q - k) - 2L(x, q) \leq M_\gamma(|h|^\gamma + |k|)$$

for all $(x, q) \in \mathbb{T}^d \times \mathbb{R}^d$ and $h, k \in \overline{B}_{R_d}$ with $R_d := \sqrt{d}/2$.

We shall see that this implies the following request on the associated Hamiltonian H , see Proposition 4.1:

(H3) there exists a constant $K = K(\gamma) > 0$ such that

$$|H(x, p) - H(y, p)| \leq K|x - y|^\gamma \quad \text{for any } x, y \in \mathbb{T}^d, p \in \mathbb{R}^d.$$

We will see that under the assumptions **(H1)**, **(H2)**, **(H3)**, there is a unique real constant α_0 for which the equation

$$-\Delta u + H(x, Du) = \alpha_0 \quad \text{in } \mathbb{T}^d, \tag{1.8}$$

admits viscosity solutions. Such solutions are actually of class C^2 and unique, up to additive constants. Our goal is to perform a discrete approximation of the solution of (1.8). In the sequel, we will use the notation $C(\mathbb{T}^d)$ to denote the family of continuous functions on \mathbb{T}^d , or, equivalently, the family of continuous and \mathbb{Z}^d -periodic functions on \mathbb{R}^d , endowed with the L^∞ -norm.

We recall some basic facts about the viscous HJ equation (1.8). Let us begin with a definition.

Definition 1.3 Let $v \in C(\mathbb{T}^d)$.

(i) We will say that v is a viscosity subsolution of (1.8) if

$$-\Delta\varphi(x_0) + H(x_0, D\varphi(x_0)) \leq \alpha_0$$

for every $\varphi \in C^2(\mathbb{T}^d)$ such that $v - \varphi$ has a local maximum at $x_0 \in \mathbb{T}^d$. Such a function φ will be called *supertangent* to v at x_0 .

(ii) We will say that v is a viscosity supersolution of (1.8) if

$$-\Delta\varphi(x_0) + H(x_0, D\varphi(x_0)) \geq \alpha_0$$

for every $\varphi \in C^2(\mathbb{T}^d)$ such that $u - \varphi$ has a local minimum at $x_0 \in \mathbb{T}^d$. Such a function φ will be called *subtangent* to v at x_0 .

We will say that v is a solution if it is both a sub and a supersolution.

Solutions, subsolutions and supersolutions will be always assumed continuous in this paper and meant in the viscosity sense, hence the term *viscosity* will be omitted in the sequel.

Remark 1.4 One gets an equivalent definition of viscosity sub and super solution by replacing, in Definition 1.3, $\varphi \in C^2(\mathbb{T}^d)$ with $\varphi \in C^\infty(\mathbb{T}^d)$ and *local maximum* or *local minimum*) by *strict local maximum* or *strict local minimum*, see for instance [2, Proposition 2.1].

Theorem 1.5 *Assume that H satisfies (H1), (H2), (H3).*

- (i) *Any Lipschitz viscosity solution u of (1.8) is of class $C^{2,\sigma}$ for any $0 < \sigma \leq \gamma$ and solves the equation in the classical sense.*
- (ii) *Classical solutions of (1.8) are unique up to additive constants.*
- (iii) *There is a unique real constant α_0 for which the equation (1.8) admits viscosity solutions.*
- (iv) *Any viscosity solution of (1.8) is Lipschitz.*

It should be noted that, in Theorem 1.5 above, the assumption (H3) is a rather strong requirement that is needed to conclude the uniqueness assertion (iii), but it is what we need in what follows.

Sketch of the proof (i) Obviously, we have $-C \leq -\Delta u \leq C$ in the viscosity sense for some constant $C > 0$, while from [10] we have $-C \leq -\Delta u \leq C$ in the viscosity sense if and only if $-C \leq -\Delta u \leq C$ in the distributional sense. Hence, $-\Delta u \in L^\infty(\mathbb{T}^d)$. Elliptic regularity theory ensures that $u \in W^{2,p}$ for any $p > 1$ and, hence, $u \in C^{1,\sigma}$ for any $0 < \sigma < 1$. Moreover, since

$$-\Delta u + H(x, Du) = \alpha \quad \text{in } \mathbb{T}^d,$$

by the Schauder theory, we have $u \in C^{2,\sigma}$ for any $0 < \sigma \leq \gamma$. (ii) Let u, v be classical solutions of (1.8). Pick $R \geq \|Du\|_\infty, \|Dv\|_\infty$ and set $C := \max_{\mathbb{T}^d \times B_{R+2}} |H(x, p)|$. By convexity, we have that $H(x, \cdot)$ is C -Lipschitz in B_R , for every $x \in \mathbb{T}^d$. Hence, by subtracting (1.8) for u and v , respectively, we get

$$\begin{aligned} 0 &= -\Delta(u - v) + H(x, Du(x)) - H(x, Dv(x)) \\ &\geq -\Delta(u - v) - C|Du(x) - Dv(x)| \quad \text{in } \mathbb{T}^d, \end{aligned}$$

that is, $w := u - v$ satisfies

$$-\Delta w - C|Dw| \leq 0 \quad \text{in } \mathbb{T}^d.$$

By the strong maximum principle, we infer that w is a constant. (iii) Observe first that the comparison result Theorem 3.3 in [7] holds for the discounted equation

$$-\Delta u + H(x, Du) + \lambda u = \alpha_0, \quad \lambda > 0, \quad (1.9)$$

and then use the argument in Sect. II of [12]. (iv) The Lipschitz regularity is a consequence of Theorem VII.1 in [11]. Indeed, if we set

$$F(x, p, A) = -\text{Tr } A + H(x, p) - \alpha_0$$

for every $(x, p) \in \mathbb{R}^{2d}$ and $d \times d$ real symmetric matrix A , then

$$|F(x, p, A) - F(y, p, A)| \leq Cd^{V/2}$$

due to the current assumption on H . This ensures that F satisfies (3.2) of [11]. The strict ellipticity (3.1) of [11] is valid with F , and thus [11, Theorem VII.1] applies to (1.8). \square

2 Discretization

Throughout this section we will assume $L : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ to satisfy condition **(L1)**, **(L2)**. We proceed to define a discrete operator $\mathcal{L}_\tau : C(\mathbb{T}^d) \rightarrow C(\mathbb{T}^d)$, where the discretization parameter τ is taken in the interval $(0, 1)$. Let us denote by $\mathcal{P}(\mathbb{T}^d)$ the set of Borel probability measures on \mathbb{T}^d endowed with the metrizable topology of weak*-convergence. The source of randomness will be the heat kernel η^τ on \mathbb{T}^d , that is the continuous function

$$\begin{aligned} \eta^\tau : \mathbb{T}^d &\rightarrow \mathcal{P}(\mathbb{R}^d) \\ y &\mapsto \eta_y^\tau, \end{aligned}$$

where η_y^τ is defined as follows:

$$\eta_y^\tau(A) := \frac{1}{(4\pi\tau)^{\frac{d}{2}}} \int_A e^{-\frac{|z-y|^2}{4\tau}} dz \quad \text{for all } A \in \mathcal{B}(\mathbb{R}^d).$$

Given $u \in C(\mathbb{T}^d)$, we have in particular

$$\int_{\mathbb{T}^d} u(z) d\eta_y^\tau(z) = \frac{1}{(4\pi\tau)^{\frac{d}{2}}} \int_{\mathbb{R}^d} u(z) e^{-\frac{|z-y|^2}{4\tau}} dz = (\eta^\tau * u)(y).$$

where $\eta^\tau(y) := (4\pi\tau)^{-\frac{d}{2}} e^{-\frac{|y|^2}{4\tau}}$.

The discrete operator $\mathcal{L}_\tau : C(\mathbb{T}^d) \rightarrow C(\mathbb{T}^d)$ is defined as follows:

$$\mathcal{L}_\tau u(x) := \min_{q \in \mathbb{R}^d} (\tau L(x, -q) + (\eta^\tau * u)(x + \tau q)) \quad \text{for every } x \in \mathbb{R}^d. \quad (2.1)$$

Proposition 2.1 *The operator $\mathcal{L}_\tau : C(\mathbb{T}^d) \rightarrow C(\mathbb{T}^d)$ is monotone and commutes with additive constants, i.e.*

- (i) $\mathcal{L}_\tau u \leq \mathcal{L}_\tau v$ in \mathbb{R}^d if $u \leq v$ in \mathbb{R}^d ;
- (ii) $\mathcal{L}_\tau(u + k) = \mathcal{L}_\tau u + k$ in \mathbb{T}^d for every $u \in C(\mathbb{T}^d)$ and $k \in \mathbb{R}$.

In particular, $\|\mathcal{L}_\tau u - \mathcal{L}_\tau v\|_\infty \leq \|u - v\|_\infty$.

Proof The fact that \mathcal{L}_τ is monotone and commutes with additive constants is apparent by its definition. Since $v - \|u - v\| \leq u \leq v + \|u - v\|$, from items (i)–(ii) we infer

$$\mathcal{L}_\tau v - \|u - v\|_\infty \leq \mathcal{L}_\tau u \leq \mathcal{L}_\tau v + \|u - v\|_\infty \quad \text{in } \mathbb{T}^d.$$

□

The following holds:

Proposition 2.2 *Let $\tau > 0$. Then there exists a constant κ_τ such that $\mathcal{L}_\tau u$ is κ_τ -Lipschitz for every $u \in C(\mathbb{T}^d)$.*

Proof For any fixed constant A , let us set

$$Q_A(x) := \{q \in \mathbb{R}^d : L(x, -q) \leq A\} \quad \text{for every } x \in \mathbb{T}^d.$$

By the growth assumptions on L , there exist constants $r(A), R(A)$ with $\lim_{A \rightarrow +\infty} r(A) = \lim_{A \rightarrow +\infty} R(A) = +\infty$ such that

$$[0, r(A)]^d \subset Q_A(x) \subset [0, R(A)]^d.$$

In particular, $Q_A(x)$ is a compact subset of \mathbb{R}^d for every $x \in \mathbb{T}^d$. Choose A_τ large enough so that the set $Q_\tau(\cdot) := Q_{A_\tau}(\cdot)$ is such that

$$Q_\tau(x) + \frac{1}{\tau} \mathbb{Z}^d = \mathbb{R}^d \quad \text{for every } x \in \mathbb{T}^d. \tag{2.2}$$

Then, for every $q \notin Q_\tau(x)$, there exists $q_x \in Q_\tau(x)$ such that $q - q_x = \mathbb{k}/\tau$ for some $\mathbb{k} \in \mathbb{Z}^d$, i.e. $\tau q = \tau q_x + \mathbb{k}$. Then, given $u \in C(\mathbb{T}^d)$, by periodicity we get

$$(\eta^\tau * u)(x + \tau q) = (\eta^\tau * u)(x + \tau q_x + \mathbb{k}) = (\eta^\tau * u)(x + \tau q_x)$$

while $L(x, -q) > A_\tau \geq L(x, q_x)$. Setting $R_\tau := R_{A_\tau}$, we have in particular

$$\mathcal{L}_\tau u(x) = \min_{q \in [0, R_\tau]^d} (\tau L(x, -q) + (\eta^\tau * u)(x + \tau q)) \quad \text{for all } x \in \mathbb{T}^d.$$

Let us denote by K_τ a Lipschitz constant of L on $\mathbb{T}^d \times [-\frac{1}{\tau}, R_\tau + \frac{1}{\tau}]^d$. Let $u \in C(\mathbb{T}^d)$ and pick $x_1, x_2 \in \mathbb{T}^d$. Let us denote by $q \in [0, R_\tau]^d$ a minimizing vector for $\mathcal{L}_\tau u(x_2)$ and set $\xi := q + (x_2 - x_1)/\tau$ so that $x_2 + \tau q = x_1 + \tau \xi$. We have

$$\begin{aligned} &\mathcal{L}_\tau u(x_1) - \mathcal{L}_\tau u(x_2) \\ &\leq \tau L(x_1, -\xi) + (\eta^\tau * u)(x_1 + \tau \xi) - \tau L(x_2, -q) - (\eta^\tau * u)(x_2 + \tau q) \\ &= \tau (L(x_1, -\xi) - L(x_2, -q)) \leq K_\tau (1 + \tau) |x_1 - x_2|. \end{aligned}$$

This gives the assertion with $\kappa_\tau := K_\tau(1 + \tau)$. □

We end this section with a result we will need in the sequel.

Proposition 2.3 For $u \in C(\mathbb{T}^d)$, $\tau > 0$ and $x \in \mathbb{R}^d$ we set

$$\operatorname{argmin}(\mathcal{L}_\tau u(x)) := \left\{ q \in \mathbb{R}^d \mid \mathcal{L}_\tau u(x) = \tau L(x, -q) + (\eta^\tau * u)(x + \tau q) \right\}$$

Then

$$-q \in \partial_p H(x, D(\eta^\tau * u)(x + \tau q)) \quad \text{for all } q \in \operatorname{argmin}(\mathcal{L}_\tau u(x)).$$

Proof Let us fix $x \in \mathbb{T}^d$. Pick a $\hat{q} \in \operatorname{argmin}(\mathcal{L}_\tau u(x))$. Then the function $q \mapsto \tau L(x, -q) + (\eta^\tau * u)(x + \tau q)$ has a minimum at \hat{q} . This implies

$$0 \in -\partial_q L(x, -\hat{q}) + D(\eta^\tau * u)(x + \tau \hat{q}),$$

or, otherwise stated,

$$D(\eta^\tau * u)(x + \tau \hat{q}) \in \partial_q L(x, -\hat{q}).$$

The assertion follows by convex duality. □

We are interested in finding solutions of the following identity

$$\mathcal{L}_\tau u = u - \tau \alpha \quad \text{in } \mathbb{T}^d, \tag{2.3}$$

where $\alpha \in \mathbb{R}$ and $u \in C(\mathbb{T}^d)$. We start with the following uniqueness result:

Theorem 2.4 There exists at most one constant $\alpha \in \mathbb{R}$ for which equation (2.3) admits solutions in $C(\mathbb{T}^d)$. Furthermore, the solution $u \in C(\mathbb{T}^d)$ of (2.3) is unique, up to additive constants.

Proof Let u_1 and u_2 be fixed points with constants α_1 and α_2 , respectively. Let x be a maximum of the difference $u_1 - u_2$. Let $q \in \mathbb{R}^d$ such that

$$u_2(x) = \tau L(x, -q) + \int_{\mathbb{T}^d} u_2(z) d\eta_y^\tau(z) + \tau \alpha_2$$

with $y := x + \tau q$. By definition we have

$$u_1(x) \leq \tau L(x, -q) + \int_{\mathbb{T}^d} u_1(z) d\eta_y^\tau(z) + \tau \alpha_1.$$

So

$$u_1(x) - u_2(x) \leq \int_{\mathbb{T}^d} (u_1(z) - u_2(z)) d\eta_y^\tau(z) + \tau(\alpha_1 - \alpha_2).$$

Since x is a maximum of $u_1 - u_2$, we get $\tau(\alpha_1 - \alpha_2) \geq 0$, hence $\alpha_1 \geq \alpha_2$. By symmetry, we obtain the equality.

Let us now assume u_1, u_2 are solutions to (2.3) for the same α . By arguing as above we get

$$u_1(x) - u_2(x) \leq \int_{\mathbb{T}^d} (u_1(z) - u_2(z)) d\eta_y^\tau(z) \leq \max_{\mathbb{T}^d} (u_1 - u_2) = u_1(x) - u_2(x),$$

hence $u_1(z) - u_2(z) = u_1(x) - u_2(x)$ for every $z \in \text{spt}(\eta_y^\tau) = \mathbb{T}^d$. □

Let us proceed to show existence.

Proof of Theorem 1 Let us denote by $\widehat{C}(\mathbb{T}^d)$ the quotient space of $C(\mathbb{T}^d)$, where we identify functions that differ by a constant, and by $\text{proj} : C(\mathbb{T}^d) \rightarrow \widehat{C}(\mathbb{T}^d)$ the projection. Since \mathcal{L}_τ commutes with the addition of constants, it defines an operator $\widehat{\mathcal{L}}_\tau : \widehat{C}(\mathbb{T}^d) \rightarrow \widehat{C}(\mathbb{T}^d)$. Let us denote by $\text{Lip}_{\kappa_\tau}(\mathbb{T}^d)$ the family of κ_τ -Lipschitz function on \mathbb{T}^d , where κ_τ is the constant provided by Proposition 2.2. The set $\widehat{\text{Lip}}_{\kappa_\tau}(\mathbb{T}^d) := \text{proj}(\text{Lip}_{\kappa_\tau}(\mathbb{T}^d))$ is a convex and compact subset of $\widehat{C}(\mathbb{T}^d)$, so we can apply Schauder fixed point Theorem (see for instance [8, Theorem 3.2, p.415]) to infer that the operator $\widehat{\mathcal{L}}_\tau : \widehat{C}(\mathbb{T}^d) \rightarrow \widehat{C}(\mathbb{T}^d)$ has a fixed-point $\widehat{u}_\tau \in \widehat{\text{Lip}}_{\kappa_\tau}(\mathbb{T}^d)$, i.e. $\widehat{\mathcal{L}}_\tau(\widehat{u}_\tau) = \widehat{u}_\tau$. Lifting these relations to $C(\mathbb{T}^d)$, we infer that there exists a constant $\alpha_\tau \in \mathbb{R}$ such that $\mathcal{L}_\tau u_\tau = u_\tau - \tau\alpha_\tau$ in \mathbb{T}^d with $u_\tau = \text{proj}^{-1}(\widehat{u}_\tau) \in \text{Lip}_{\kappa_\tau}(\mathbb{T}^d)$. The asserted uniqueness of α_τ in \mathbb{R} and u_τ in $C(\mathbb{T}^d)$ is guaranteed by Theorem 2.4. □

In view that discretization is often associated with numerical computations, we give another proof of Theorem 1, which relies on Banach’s fixed point theorem instead of Schauder’s fixed point theorem. The argument we use is also known in the PDE literature as *ergodic approximation*.

Second proof of Theorem 1 Let $\delta > 0$, and consider the discounted problem

$$(1 + \delta)u - \mathcal{L}_\tau u = 0 \text{ in } \mathbb{T}^d.$$

By Proposition 2.1, $\mathcal{L}_\tau : C(\mathbb{T}^d) \rightarrow C(\mathbb{T}^d)$ is 1-Lipschitz. Hence, by Banach’s fixed point theorem, $(1 + \delta)^{-1}\mathcal{L}_\tau$ has a unique fixed point $v^\delta \in C(\mathbb{T}^d)$, which is a unique solution of $(1 + \delta)u - \mathcal{L}_\tau u = 0$ in \mathbb{T}^d . Let $\kappa_\tau > 0$ be the constant given by Proposition 2.2, so that $v^\delta = (1 + \delta)^{-1}\mathcal{L}_\tau v^\delta$ is $(1 + \delta)^{-1}\kappa_\tau$ -Lipschitz on \mathbb{T}^n . Accordingly, the family $\{v^\delta \mid \delta > 0\}$ is equi-Lipschitz on \mathbb{T}^d . By the Ascoli-Arzelà theorem, we can select a sequence of $\delta_j > 0$ converging to zero such that the functions $v^{\delta_j} - \min_{\mathbb{T}^d} v^{\delta_j}$ converge to a function w in $C(\mathbb{T}^d)$ as $j \rightarrow \infty$. Setting $m_j := \min_{\mathbb{T}^d} v^{\delta_j}$ and $w_j := v^{\delta_j} - m_j$, we observe by Proposition 2.1 that

$$0 = (1 + \delta_j)(w_j + m_j) - \mathcal{L}_\tau(w_j + m_j) = (1 + \delta_j)w_j + \delta_j m_j - \mathcal{L}_\tau w_j,$$

where the first and last terms in the last expression converge to w and $\mathcal{L}_\tau w$ in $C(\mathbb{T}^d)$, respectively. Consequently, the sequence of the constants $\delta_j m_j$ converges to a constant $-\tau\alpha_\tau$, which implies that $\mathcal{L}_\tau w - w = \tau\alpha_\tau$ in \mathbb{T}^d . □

The standard proof of Banach’s fixed point theorem is constructive or iterative, and therefore, the above proof can be easily implemented for numerical computations.

Definition 2.5 We say that $u \in C(\mathbb{T}^d)$ is an α -subsolution for \mathcal{L}_τ if

$$\mathcal{L}_\tau u \geq u - \tau\alpha \quad \text{in } \mathbb{T}^d.$$

Denote by $\mathcal{H}_\tau(\alpha)$ the set of α -subsolutions.

By taking into account the properties of the Lax operator stated in Proposition 2.1, we easily infer the following facts:

Proposition 2.6 *The sets $\mathcal{H}_\tau(\alpha)$ are convex and closed subset of $C(\mathbb{T}^d)$ and increasing with respect to α , i.e. $\mathcal{H}_\tau(\alpha) \subseteq \mathcal{H}_\tau(\beta)$ if $\alpha \leq \beta$. Furthermore:*

- (i) $u + k \in \mathcal{H}_\tau(\alpha)$ for every $u \in \mathcal{H}_\tau(\alpha)$ and $k \in \mathbb{R}$;
- (ii) $\mathcal{L}_\tau(\mathcal{H}_\tau(\alpha)) \subseteq \mathcal{H}_\tau(\alpha)$.

Next, we show that all α_τ -subsolutions for \mathcal{L}_τ are actually solutions to (5).

Proposition 2.7 *Let $(\alpha_\tau, u_\tau) \in \mathbb{R} \times C(\mathbb{T}^d)$ be a solution of*

$$\mathcal{L}_\tau u_\tau = u_\tau - \tau\alpha_\tau \quad \text{in } \mathbb{T}^d.$$

Then $\mathcal{H}_\tau(\alpha_\tau) = \{u_\tau + k : k \in \mathbb{R}\}$. Furthermore,

$$\alpha_\tau = \min\{\alpha : \mathcal{H}_\tau(\alpha) \neq \emptyset\}. \tag{2.4}$$

Proof Let us pick $u \in \mathcal{H}_\tau(\alpha)$ and argue as in the proof of Theorem 2.4 with $u_1 := u$ and $u_2 := u_\tau$. By also using the fact that $u \leq \mathcal{L}_\tau u + \tau\alpha$, we end up with

$$u(x) - u_\tau(x) \leq \int_{\mathbb{T}^d} (u(z) - u_\tau(z)) d\eta_y(z) + \tau(\alpha - \alpha_\tau) \quad \text{for all } x \in \mathbb{T}^d.$$

By picking x as a maximum point of $u - u_\tau$, we conclude that $\alpha \geq \alpha_\tau$. When $\alpha = \alpha_\tau$, we furthermore get that $u - u_\tau$ is constant. □

We conclude this section by deriving the following bounds on the constant α_τ .

Proposition 2.8 *The following holds:*

$$-\max_{y \in \mathbb{T}^d} \min_{q \in \mathbb{R}^d} L(y, q) \leq \alpha_\tau \leq -\min_{\mathbb{T}^d \times \mathbb{R}^d} L.$$

Proof Let us set $L_m(x) := \min_{q \in \mathbb{R}^d} L(x, q)$. Pick $u \in \mathcal{H}_\tau(\alpha_\tau)$ and set $v(x) := \max_{\mathbb{T}^d} u$, $w(x) := \min_{\mathbb{T}^d} u$ for all $x \in \mathbb{R}^d$. Then, for all $x \in \mathbb{T}^d$,

$$u(x) - \tau\alpha_\tau = \mathcal{L}_\tau u(x) \leq \mathcal{L}_\tau v(x) = \tau L_m(x) + \max_{\mathbb{T}^d} u \leq \tau \max_{\mathbb{T}^d} L_m + \max_{\mathbb{T}^d} u$$

and

$$u(x) - \tau\alpha_\tau = \mathcal{L}_\tau u(x) \geq \mathcal{L}_\tau w(x) = \tau L_m(x) + \min_{\mathbb{T}^d} u \geq \tau \min_{\mathbb{T}^d} L_m + \min_{\mathbb{T}^d} u.$$

That implies $-\max_{\mathbb{T}^d} L_m \leq \alpha_\tau \leq -\min_{\mathbb{T}^d} L_m$, as it was asserted. \square

3 Equi-continuity of the functions u_τ

This section is devoted to prove equi-continuity of the functions $\{u_\tau : \tau \in (0, 1)\}$, where u_τ denotes a solution in $C(\mathbb{T}^d)$ of the equation

$$\mathcal{L}_\tau u_\tau = u_\tau - \tau\alpha_\tau \quad \text{in } \mathbb{T}^d \quad (3.1)$$

and α_τ is the constant given by Theorem 1. Throughout the rest of the paper, we will assume that L satisfies the following further condition, for some constants $\gamma \in (0, 1)$ and $M_\gamma > 0$:

$$L(x+h, q+k) + L(x-h, q-k) - 2L(x, q) \leq M_\gamma(|h|^\gamma + |k|) \quad (\mathbf{L3})$$

for all $(x, q) \in \mathbb{T}^d \times \mathbb{R}^d$ and $h, k \in B_{R_d}$ with $R_d := \sqrt{d}/2$.

We start by noticing that

$$\begin{aligned} \mathcal{L}_\tau u(x) &= \min_{q \in \mathbb{R}^d} \left(\tau L\left(x, \frac{q}{\tau}\right) + (\eta^\tau * u)(x - q) \right) \\ &= \min_{y \in \mathbb{R}^d} \left(\tau L\left(x, \frac{x - y}{\tau}\right) + (\eta^\tau * u)(y) \right). \end{aligned}$$

We introduce the operators $\mathcal{E}_\tau, \mathcal{F}_\tau : C(\mathbb{T}^d) \rightarrow C(\mathbb{T}^d)$ defined as

$$\mathcal{E}_\tau u = \eta^\tau * u, \quad \mathcal{F}_\tau u(x) = \min_{y \in \mathbb{R}^d} \left(\tau L\left(x, \frac{x - y}{\tau}\right) + u(y) \right).$$

Notice that $\mathcal{L}_\tau = \mathcal{F}_\tau \circ \mathcal{E}_\tau$. For $u \in C(\mathbb{T}^d)$ and $\sigma \in (0, 2]$, we set

$$\begin{aligned} \Theta_\sigma(u) &:= \inf\{a \geq 0 \mid u(x+h) + u(x-h) - 2u(x) \\ &\leq a|h|^\sigma \quad \text{for all } x, h \in \mathbb{R}^d\}, \end{aligned}$$

and

$$\Lambda_\sigma(\mathbb{T}^d) = \{v \in C(\mathbb{T}^d) \mid \Theta_\sigma(v) < \infty\}.$$

We also introduced the following temporary notation, defined for $R > 0$:

$$\Theta_{\sigma,R}(u) := \inf\{a \geq 0 \mid u(x+h) + u(x-h) - 2u(x) \leq a|h|^\sigma \text{ for } x, h \in \mathbb{R}^d, \text{ with } |h| \leq R\}.$$

We start with some preliminary results.

Lemma 3.1 *Let $u \in \Lambda_\sigma(\mathbb{T}^d)$, with $\sigma \in (0, 2]$, and $R \geq R_d := \sqrt{d}/2$. Then*

$$\Theta_{\sigma,R}(u) = \Theta_\sigma(u).$$

Proof It is obvious that

$$\Theta_{\sigma,R}(u) \leq \Theta_\sigma(u). \tag{3.2}$$

To prove the reversed inequality, we fix $x \in \mathbb{R}^d$ and $a \geq 0$ and assume that

$$u(x+h) + u(x-h) - 2u(x) \leq a|h|^\sigma \text{ for } x, h \in \mathbb{R}^d, \text{ with } |h| \leq R. \tag{3.3}$$

Set

$$f(h) := u(x+h) + u(x-h) - 2u(x) \text{ for all } h \in \mathbb{R}^d,$$

and observe that $f \in C(\mathbb{T}^d)$. By the periodicity of f , we see that

$$M := \max_{\mathbb{R}^d} f = \max\{f(h) \mid h \in \mathbb{R}^d, |h_i| \leq 1/2 \text{ for } i = 1, \dots, d\}.$$

Since $f(0) = 0$, we have $M \geq 0$.

By (3.3), we have

$$\begin{aligned} M &\leq a \max\{|h|^\sigma \mid h \in \mathbb{R}^d, |h_i| \leq 1/2 \text{ for } i = 1, \dots, d\} \\ &\leq a \max\{|h|^\sigma \mid h \in \mathbb{R}^d, |h| \leq R\} = aR^\sigma, \end{aligned}$$

which implies that

$$f(h) \leq M \leq aR^\sigma \leq a|h|^\sigma \text{ for } h \in \mathbb{R}^d \setminus B_R.$$

This together with (3.3) yields

$$f(h) \leq a|h|^\sigma \text{ for all } h \in \mathbb{R}^d.$$

Thus, we have the reversed inequality of (3.2) and conclude that $\Theta_\sigma(u) = \Theta_{\sigma,R}(u)$. □

We derive the following consequence.

Corollary 3.2 *Let $0 < \rho < \sigma \leq 2$. Then*

$$\Lambda_\sigma(\mathbb{T}^d) \subset \Lambda_\rho(\mathbb{T}^d).$$

Proof Let $u \in \Lambda_\sigma(\mathbb{T}^d)$ and $x, h \in \mathbb{R}^d$. If $|h| \leq R_d$, then we have

$$u(x+h) + u(x-h) - 2u(x) \leq \Theta_\sigma(u)|h|^\sigma \leq \Theta_\sigma(u)R_d^{\sigma-\rho}|h|^\rho,$$

which shows that $\Theta_{\rho,R_d}(u) < \infty$. By Lemma 3.1, we see that $\Theta_\rho(u) < \infty$ and, moreover, that $\Lambda_\sigma(\mathbb{T}^d) \subset \Lambda_\rho(\mathbb{T}^d)$. \square

For $\sigma \in (0, 1]$ and $u \in C(\mathbb{R}^d)$, we write

$$[u]_\sigma := \sup_{x,y \in \mathbb{R}^d, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\sigma},$$

and

$$C^{0,\sigma}(\mathbb{T}^d) := \left\{ u \in C(\mathbb{T}^d) \mid [u]_\sigma < \infty \right\}.$$

The following lemma is similar to Corollary 3.2.

Lemma 3.3 *Let $0 < \rho < \sigma \leq 1$. Then*

$$C^{0,\sigma}(\mathbb{T}^d) \subset C^{0,\rho}(\mathbb{T}^d).$$

Proof Let $u \in C^{0,\sigma}(\mathbb{T}^d)$. For any $x, y \in \mathbb{R}^d$, we choose $z \in \mathbb{Z}^d$ so that

$$|x - y - z| = \min_{\zeta \in \mathbb{Z}^d} |x - y - \zeta|,$$

and note that

$$z \in \prod_{i=1}^d [x_i - y_i - 1/2, x_i - y_i + 1/2],$$

and

$$\begin{aligned} |u(x) - u(y)| &= |u(x) - u(y+z)| \leq [u]_\sigma |x - y - z|^\sigma \\ &\leq [u]_\sigma R_d^{\sigma-\rho} |x - y - z|^\rho \leq [u]_\sigma R_d^{\sigma-\rho} |x - y|^\rho. \end{aligned}$$

This shows that $C^{0,\sigma}(\mathbb{T}^d) \subset C^{0,\rho}(\mathbb{T}^d)$. \square

Proposition 3.4 *For any $\sigma \in (0, 1)$, we have*

$$\Lambda_\sigma(\mathbb{T}^d) = C^{0,\sigma}(\mathbb{T}^d)$$

and for some constant $C_\sigma > 1$, depending only on σ ,

$$C_\sigma^{-1} \Theta_\sigma(u) \leq [u]_\sigma \leq C_\sigma \Theta_\sigma(u) \quad \text{for } u \in \Lambda_\sigma(\mathbb{T}^d).$$

Proof Let $u \in C^{0,\sigma}(\mathbb{T}^d)$ and $x, h \in \mathbb{R}^d$. Compute that

$$\begin{aligned} u(x+h) + u(x-h) - 2u(x) &= u(x+h) - u(x) + u(x-h) - u(x) \\ &\leq [u]_\sigma |h|^\sigma + [u]_\sigma |h|^\sigma = 2[u]_\sigma |h|^\sigma, \end{aligned}$$

which shows that

$$\Theta_\sigma(u) \leq 2[u]_\sigma,$$

and that $C^{0,\sigma}(\mathbb{T}^d) \subset \Lambda_\sigma(\mathbb{T}^d)$.

Now, let $u \in \Lambda_\sigma(\mathbb{T}^d)$, so that we have

$$u(x) \geq \frac{1}{2}(u(x+h) + u(x-h) - \Theta_\sigma(u)|h|^\sigma) \quad \text{for all } x, h \in \mathbb{R}^d. \tag{3.4}$$

Let $x, y \in \mathbb{R}^d$. We intend to show that

$$u(x) - u(y) \leq C|x - y|^\sigma$$

for some constant $C > 0$, depending only on $\Theta_\sigma(u)$ and σ .

By translation, we may assume that $x = 0$. We need to show that for some constant $C > 0$,

$$u(y) \geq u(0) - C|y|^\sigma. \tag{3.5}$$

We may assume that $y \neq 0$. Set $K = \Theta_\sigma(u)$ and let $m \in \mathbb{N}$ large enough. By (3.4), we have

$$u(2^{k-1}y) \geq \frac{1}{2}(u(0) + u(2^k y) - K|2^{k-1}y|^\sigma) \quad \text{for } k = 1, \dots, m.$$

From these, we obtain

$$\begin{aligned} \sum_{k=1}^m 2^{-(k-1)} u(2^{k-1}y) &\geq \sum_{k=1}^m 2^{-k} [u(0) + u(2^k y) - K2^{\sigma(k-1)}|y|^\sigma] \\ &= \frac{2^m - 1}{2^m} u(0) + \sum_{k=1}^m 2^{-k} u(2^k y) - K2^{-\sigma} \sum_{k=1}^m 2^{(\sigma-1)k} |y|^\sigma. \end{aligned}$$

After rewriting the first summation above as $u(y) + \sum_{k=1}^{m-1} 2^{-k} u(2^k y)$, we get

$$\begin{aligned} u(y) &\geq (1 - 2^{-m})u(0) + 2^{-m}u(2^m y) - K2^{-\sigma}2^{\sigma-1} \frac{1 - 2^{(\sigma-1)m}}{1 - 2^{\sigma-1}} |y|^\sigma \\ &\geq u(0) + 2^{-m}(u(2^m y) - u(0)) - \frac{K}{2 - 2^\sigma} |y|^\sigma \\ &\geq u(0) - 2^{-m} \operatorname{osc}(u) - \frac{K}{2 - 2^\sigma} |y|^\sigma, \end{aligned}$$

where $\operatorname{osc}(u) := \max_{\mathbb{T}^d} u - \min_{\mathbb{T}^d} u$.

Sending $m \rightarrow +\infty$ yields

$$u(y) \geq u(0) - \frac{\Theta_\sigma(u)}{2 - 2^\sigma} |y|^\sigma,$$

which proves (3.5), with $C = \Theta_\sigma(u)/(2 - 2^\sigma)$. This readily shows that

$$|u(x) - u(y)| \leq \frac{\Theta_\sigma(u)}{2 - 2^\sigma} |x - y|^\sigma \quad \text{for all } x, y \in \mathbb{R}^d.$$

Hence, we have

$$[u]_\sigma \leq \frac{1}{2 - 2^\sigma} \Theta_\sigma(u) \quad \text{and hence} \quad \Lambda_\sigma(\mathbb{T}^d) \subset C^{0,\sigma}(\mathbb{T}^d).$$

The proof is now complete. □

One can show that if $\sigma \in (1, 2]$, then $\Lambda_\sigma(\mathbb{T}^d) \subset C^{0,1}(\mathbb{T}^d)$, which is left to the interested reader to check.

Let us now prove the equi-continuity of the functions $\{u_\tau : \tau \in (0, 1)\}$. We start with the following result:

Theorem 3.5 *Let $u \in \Lambda_\gamma(\mathbb{T}^d)$. Then*

$$\Theta_\gamma(\mathcal{F}_\tau u) \leq \frac{1}{(1 + \tau)^\gamma} \Theta_\gamma(u) + \tau M_\gamma \left(1 + \frac{R_d^{1-\gamma}}{1 + \tau} \right).$$

Proof Let $x \in \mathbb{R}^d$ and choose $y \in \mathbb{R}^d$ so that

$$\mathcal{F}_\tau u(x) = u(y) + \tau L\left(x, \frac{x - y}{\tau}\right).$$

For every $h, k \in \mathbb{R}^d$ we get

$$\begin{aligned} & \mathcal{F}_\tau u(x+h) + \mathcal{F}_\tau u(x-h) - 2\mathcal{F}_\tau u(x) \\ & \leq u(y+k) + \tau L\left(x+h, \frac{x+h-(y+k)}{\tau}\right) + u(y-k) \\ & \quad + \tau L\left(x-h, \frac{x-h-(y-k)}{\tau}\right) \\ & \quad - 2u(y) - 2\tau L\left(x, \frac{x-y}{\tau}\right) \\ & \leq \Theta_\gamma(u)|k|^\gamma + \tau \left[L\left(x+h, \frac{x-y}{\tau} + \frac{h-k}{\tau}\right) \right. \\ & \quad \left. + L\left(x-h, \frac{x-y}{\tau} - \frac{h-k}{\tau}\right) - 2\tau L\left(x, \frac{x-y}{\tau}\right) \right] \end{aligned}$$

In order to exploit **(L3)**, we take $h \in \bar{B}_{R_d}$ and $k := h/(1 + \tau)$, so that $|\frac{h-k}{\tau}| = |\frac{h}{1+\tau}| < R_d$. We infer

$$\mathcal{F}_\tau u(x+h) + \mathcal{F}_\tau u(x-h) - 2\mathcal{F}_\tau u(x) \leq \Theta_\gamma(u) \frac{|h|^\gamma}{(1+\tau)^\gamma} + \tau M_\gamma \left(|h|^\gamma + \frac{|h|}{1+\tau} \right),$$

which implies

$$\Theta_{\gamma, R_d}(\mathcal{F}_\tau u) \leq \frac{1}{(1+\tau)^\gamma} \Theta_\gamma(u) + \tau M_\gamma \left(1 + \frac{R_d^{1-\gamma}}{1+\tau} \right).$$

The assertion follows in view of Lemma 3.1. □

Lemma 3.6 *Let $u \in \Lambda_\sigma(\mathbb{T}^d)$, with $\sigma \in (0, 2]$. We have*

$$\Theta_\sigma(\mathcal{E}_\tau u) \leq \Theta_\sigma(u).$$

Proof Let $x, h \in \mathbb{R}^d$. We compute that

$$\begin{aligned} & \mathcal{E}_\tau u(x+h) + \mathcal{E}_\tau u(x-h) - 2\mathcal{E}_\tau u(x) \\ & = \int_{\mathbb{R}^d} \eta^\tau(y) \left(u(x-y+h) + u(x-y-h) - 2u(x-y) \right) dy \\ & \leq \int_{\mathbb{R}^d} \eta^\tau(z) \Theta_\sigma(u) |h|^\sigma dz = \Theta_\sigma(u) |h|^\sigma, \end{aligned}$$

which yields

$$\Theta_\sigma(\mathcal{E}_\tau u) \leq \Theta_\sigma(u).$$

□

Theorem 3.7 *Let $u \in \Lambda_\gamma(\mathbb{T}^d)$. Then*

$$\Theta_\gamma(\mathcal{L}_\tau u) \leq \frac{1}{(1 + \tau)^\gamma} \Theta_\gamma(u) + \tau M_\gamma \left(1 + \frac{R_d^{1-\gamma}}{1 + \tau} \right).$$

Proof By Theorem 3.5 and Lemma 3.6, we obtain

$$\begin{aligned} \Theta_\gamma(\mathcal{L}_\tau u) &= \Theta_\gamma(\mathcal{F}_\tau \circ \mathcal{E}_\tau u) \\ &\leq \frac{1}{(1 + \tau)^\gamma} \Theta_\gamma(\mathcal{E}_\tau u) + \tau M_\gamma \left(1 + \frac{R_d^{1-\gamma}}{1 + \tau} \right) \\ &\leq \frac{1}{(1 + \tau)^\gamma} \Theta_\gamma(u) + \tau M_\gamma \left(1 + \frac{R_d^{1-\gamma}}{1 + \tau} \right). \end{aligned}$$

□

Lemma 3.8 *Let $u \in C(\mathbb{T}^d)$ and $\sigma \in (0, 2]$. Then*

$$\mathcal{E}_\tau u \in \Lambda_\sigma(\mathbb{T}^d).$$

Proof Set

$$v_\tau(x) = \mathcal{E}_\tau u(x) \quad \text{for } x \in \mathbb{T}^d.$$

As is well-known (and easily shown), the function v_τ is smooth and periodic in \mathbb{R}^d . In particular, the second derivatives of v_τ are bounded in \mathbb{R}^d , which implies that v_τ is semi-concave in \mathbb{R}^d , that is, $\Theta_2(\mathcal{E}_\tau u) < \infty$. Thus, we find that $\mathcal{E}_\tau u \in \Lambda_2(\mathbb{T}^d)$ and, due to Corollary 3.2, that $\mathcal{E}_\tau u \in \Lambda_\sigma(\mathbb{T}^d)$ for all $\sigma \in (0, 2]$. □

Theorem 3.9 *Let $\tau > 0$ and $u_\tau \in C(\mathbb{T}^d)$ satisfy (3.1). Then*

$$\Theta_\gamma(u_\tau) \leq \frac{\tau B_\tau}{1 - A_\tau},$$

where

$$A_\tau = \frac{1}{(1 + \tau)^\gamma}, \quad B_\tau = M_\gamma \left(1 + \frac{R_d^{1-\gamma}}{1 + \tau} \right) \quad \text{and} \quad R_d = \frac{\sqrt{d}}{2}.$$

We remark that, in the theorem above, $0 < A_\tau < 1$,

$$\lim_{\tau \rightarrow 0} \frac{\tau}{1 - A_\tau} = \frac{1}{\gamma},$$

and for any $0 < T < \infty$,

$$\sup_{0 < \tau \leq T} \frac{\tau B_\tau}{1 - A_\tau} < \infty.$$

Proof Using Lemma 3.8 and Theorem 3.5, we infer from (3.1) that

$$u_\tau = \mathcal{F}_\tau \circ \mathcal{E}_\tau u_\tau + \tau \alpha_\tau \in \Lambda_\gamma(\mathbb{T}^d).$$

By (3.1) and Theorem 3.7, we get

$$\Theta_\gamma(u_\tau) = \Theta_\gamma(u_\tau - \tau \alpha_\tau) = \Theta_\gamma(\mathcal{L}_\tau u_\tau) \leq A_\tau \Theta_\gamma(u_\tau) + \tau B_\tau,$$

from which follows

$$\Theta_\gamma(u_\tau) \leq \frac{\tau B_\tau}{1 - A_\tau}.$$

□

As a consequence of the information gathered, we derive the following fact:

Proposition 3.10 *The family of functions $\{u_\tau \mid \tau \in (0, 1)\}$ is equi-continuous on \mathbb{T}^d .*

Proof It is a direct consequence of Theorem 3.9 and Proposition 3.4

□

4 The approximation result

4.1 Proof of Theorem 2.

This section is devoted to the proof of Theorem 2. We begin by showing that, under assumptions (L1), (L2), (L3) on the Lagrangian L , Theorem 1.5 applies.

Proposition 4.1 *Assume that L satisfies (L1), (L2), (L3). Then the associated Hamiltonian H satisfies conditions (H1), (H2), (H3).*

Proof The fact that H satisfies (H1), (H2), i.e. it is convex and superlinear, is standard. Let us prove (H3). For each $q \in \mathbb{R}^d$ the function $u_q(x) = L(x, q)$ belongs to $\Lambda_\gamma(\mathbb{T}^d)$ with $\Theta_\gamma(u_q) \leq M_\gamma$. From Proposition 3.4 we get $u_q \in C^{0,\gamma}(\mathbb{T}^d)$ and $[u_q]_\gamma \leq C_\gamma M_\gamma := D_\gamma$. Thus

$$|L(x, q) - L(y, q)| \leq D_\gamma |x - y|^\gamma \text{ for any } x, y \in \mathbb{T}^d, q \in \mathbb{R}^d.$$

For $x, p, h \in \mathbb{R}^d$, let q_\pm be such that

$$H(x \pm h, p) = pq_\pm - L(x, q_\pm)$$

and let $v = \frac{1}{2}(q_+ + q_-)$, $u = \frac{1}{2}(q_+ - q_-)$, so that $q_{\pm} = v \pm u$. Then

$$\begin{aligned} & H(x + h, p) + H(x - h, p) - 2H(x, p) \\ & \leq p(v + u) + p(v - u) - 2pv - L(x + h, v + u) - L(x - h, v - u) + 2L(x, v) \\ & = L(x, v + u) - L(x + h, v + u) + L(x, v - u) - L(x - h, v - u) \\ & \quad - L(x, v + u) - L(x, v - u) + 2L(x, v) \\ & \leq 2D_{\gamma} |h|^{\gamma}, \end{aligned}$$

where, for the last inequality, we have also exploited the convexity of $L(x, \cdot)$. Thus, for each $p \in \mathbb{R}^d$, the function $w_p(x) = H(x, p)$ belongs to $\Lambda_{\gamma}(\mathbb{T}^d)$ with $\Theta_{\gamma}(w_p) \leq 2D_{\gamma}$, so we have that $w_p \in C^{0,\gamma}(\mathbb{T}^d)$ with $[w_p]_{\gamma} \leq 2C_{\gamma}D_{\gamma} = 2M_{\gamma}C_{\gamma}^2$. \square

Next, we prove an auxiliary lemma.

Lemma 4.2 *Let $\varphi \in C^2(\mathbb{T}^d)$. For every $R > 0$, there exists a continuous function $\omega : [0, +\infty) \rightarrow [0, +\infty)$ vanishing at 0, only depending on R and φ , such that*

$$\left| \frac{(\eta^{\tau} * \varphi)(x + \tau q) - \varphi(x)}{\tau} - \langle D\varphi(x), q \rangle - \Delta\varphi(x) \right| \leq \omega(\tau)$$

for all $(x, q) \in \mathbb{T}^d \times B_R$ and $\tau > 0$.

Proof Let $\varphi \in C^2(\mathbb{T}^d)$ and set $u(x, t) = \eta^t * \varphi(x)$ for $(x, t) \in \mathbb{R}^d \times (0 + \infty)$ and $u(x, 0) = \varphi(x)$ for $x \in \mathbb{R}^d$. It is a standard observation that for any multi-indices α , with $|\alpha| \leq 2$, $D_x^{\alpha}u \in C(\mathbb{R}^d \times [0 + \infty))$ and $D_x^{\alpha}u(x, 0) = D^{\alpha}\varphi(x)$ for all $x \in \mathbb{R}^d$, and that $\partial u / \partial t - \Delta u = 0$ in $\mathbb{R}^d \times (0, +\infty)$. Observe that, for any $(x, q) \in \mathbb{R}^d \times B_R$,

$$\begin{aligned} \eta^{\tau} * \varphi(x + \tau q) - \varphi(x) &= \int_0^{\tau} \frac{du}{dt}(x + tq, t) dt \\ &= \int_0^{\tau} \left(\langle D_x u(x + tq, t), q \rangle + \frac{\partial u}{\partial t}(x + tq, t) \right) dt \\ &= \int_0^{\tau} \left(\langle D_x u(x + tq, t), q \rangle + \Delta u(x + tq, t) \right) dt. \end{aligned}$$

Now, setting

$$\omega(r) = \max_{(x,q,t) \in \mathbb{R}^d \times B_R \times [0,r]} |\langle D_x u(x + tq, t), q \rangle + \Delta u(x + tq, t) - \Delta\varphi(x)|,$$

we have

$$\omega \in C([0 + \infty)), \quad \omega(0) = 0,$$

and

$$\left| \frac{(\eta^\tau * \varphi)(x + \tau q) - \varphi(x)}{\tau} - \langle D\varphi(x), q \rangle - \Delta\varphi(x) \right| \leq \omega(\tau)$$

for all $(x, q) \in \mathbb{T}^d \times B_R$ and $\tau > 0$. □

Proof of Theorem 2 It follows from Proposition 4.1 and Theorem 1.5 that there exists a unique pair $(\alpha, u) \in \mathbb{R} \times C(\mathbb{T}^d)$ with $u(x_0) = 0$ such that u is a viscosity solution to

$$-\Delta u + H(x, Du) = \alpha \quad \text{in } \mathbb{T}^d. \tag{4.1}$$

In view of Proposition 2.8 and Proposition 3.10, of the fact that $u_\tau(x_0) = 0$ for all $\tau \in (0, 1)$ and of Arzelà-Ascoli Theorem, we have that the set $\{(\alpha_\tau, u_\tau) \mid \tau \in (0, 1)\}$ is precompact in $\mathbb{R} \times C(\mathbb{T}^d)$. In order to prove the assertion, it is therefore enough to show that, if the pair (α, u) is the limit of $(\alpha_{\tau_n}, u_{\tau_n})$ in $\mathbb{R} \times C(\mathbb{T}^d)$ for some $\tau_n \rightarrow 0$, then u is a solution to (4.1). Let us first show that such an u is a viscosity subsolution to (4.1). Let $\varphi \in C^3(\mathbb{T}^d)$ be such that $u - \varphi$ has a strict maximum at x_0 . Then there exists a sequence of points $(x_n)_n$ converging to x_0 in \mathbb{T}^d such that $u_{\tau_n} - \varphi$ has a maximum at x_n . Let us set $\varepsilon_n := \max(u_{\tau_n} - \varphi)$ and $\varphi_n := \varphi + \varepsilon_n$. Then

$$u_{\tau_n} \leq \varphi_n \quad \text{in } \mathbb{T}^d \quad \text{and} \quad u_{\tau_n}(x_n) = \varphi_n(x_n).$$

By the monotone character of the operator \mathcal{L}_{τ_n} we infer

$$\varphi_n(x_n) = u_{\tau_n}(x_n) = \mathcal{L}_{\tau_n} u_{\tau_n}(x_n) + \tau_n \alpha_{\tau_n} \leq \mathcal{L}_{\tau_n} \varphi_n(x_n) + \tau_n \alpha_{\tau_n},$$

hence, since $\varphi_n = \varphi + \varepsilon_n$,

$$\frac{\varphi(x_n) - \mathcal{L}_{\tau_n} \varphi(x_n)}{\tau_n} \leq \alpha_{\tau_n} \tag{4.2}$$

By definition of \mathcal{L}_{τ_n} , we infer that, for every fixed $q \in \mathbb{R}^d$,

$$\frac{\varphi(x_n) - (\eta^{\tau_n} * \varphi)(x_n + \tau_n q)}{\tau_n} - L(x_n, -q) \leq \alpha_{\tau_n}$$

By sending $n \rightarrow +\infty$ and by making use of Lemma 4.2, we end up with

$$-\Delta\varphi(x_0) + \langle D\varphi(x_0), -q \rangle - L(x_0, -q) \leq \alpha.$$

By taking the supremum of the above inequality with respect to $q \in \mathbb{R}^d$, we finally get, by the duality between L and H ,

$$-\Delta\varphi(x_0) + H(x_0, D\varphi(x_0)) \leq \alpha,$$

thus showing that u is a viscosity subsolution to (4.1). Let us now show that u is a viscosity supersolution to (4.1). Let $\varphi \in C^3(\mathbb{T}^d)$ be such that $u - \varphi$ has a strict

minimum at x_0 . Then there exists a sequence of points $(x_n)_n$ converging to x_0 in \mathbb{T}^d such that $u_{\tau_n} - \varphi$ has a minimum at x_n . Let us set $\varepsilon_n := \min(u_{\tau_n} - \varphi)$ and $\varphi_n := \varphi + \varepsilon_n$. Then

$$u_{\tau_n} \geq \varphi_n \quad \text{in } \mathbb{T}^d \quad \text{and} \quad u_{\tau_n}(x_n) = \varphi_n(x_n).$$

By arguing analogously, we end up with

$$\frac{\varphi(x_n) - \mathcal{L}_{\tau_n} \varphi(x_n)}{\tau_n} \geq \alpha_{\tau_n}$$

For each $n \in \mathbb{N}$, pick a minimizing $q_n \in \mathbb{R}^d$ for $\mathcal{L}_{\tau_n} \varphi(x_n)$, so that the previous inequality rereads as

$$\frac{\varphi(x_n) - (\eta^{\tau_n} * \varphi)(x_n + \tau_n q_n)}{\tau_n} - L(x_n, -q_n) \geq \alpha_{\tau_n}. \tag{4.3}$$

By making use of Proposition 2.3 and of the fact that $|D(\eta^\tau * \varphi)(x_n)| \leq \|D\varphi\|_\infty$, we infer that there exists $R > 0$ such that $q_n \in B_R$ for every $n \in \mathbb{N}$. Up to extracting a further subsequence if necessary, we can assume that $q_n \rightarrow q$. Now we send $n \rightarrow +\infty$ in (4.3) to get

$$-\Delta\varphi(x_0) + \langle D\varphi(x_0), -q \rangle - L(x_0, -q) \geq \alpha.$$

By the duality between L and H , this implies

$$-\Delta\varphi(x_0) + H(x_0, D\varphi(x_0)) \geq \alpha,$$

finally showing that u is a viscosity supersolution to (4.1). □

4.2 Examples

In this section, we exhibit some examples of Lagrangians for which the conclusion of Theorem 2 holds true.

Example 1 $L \in C(\mathbb{T}^d \times \mathbb{R}^d)$ satisfies (L1), (L2) and $\min_{q \in \mathbb{R}^d} L(x, q) = c$ for all $x \in \mathbb{T}^d$ for some constant $c \in \mathbb{R}$.

This example includes the case when L is independent of x , or the case $L(x, q) = a(x)|q|^m$ with $m \in (1, +\infty)$ and $a : \mathbb{T}^d \rightarrow (0, +\infty)$ Lipschitz continuous.

In this case $\alpha_\tau = -c$ by Proposition 2.8 and $u_\tau \equiv 0$ for every $\tau > 0$, so convergence of the u_τ trivially holds.

Example 2 $L(x, q) = L_0(q) + f(x)$ where $f \in C^{0,\gamma}(\mathbb{T}^d)$ with $\gamma \in (0, 1)$ and $L_0 \in C(\mathbb{R}^d)$ satisfies (L1), (L2) and

$$D_q^2 L_0(q) \leq C_0 I_d \quad \text{in } \mathbb{R}^d \setminus \overline{B}_{R_0} \tag{4.4}$$

in the sense of distributions, for some constants $C_0 > 0$ and $R_0 > 0$.

This example includes the case $L_0(q) := |q|^m$ with $m \in (1, 2]$. Indeed,

$$DL_0(q) = m|q|^{m-2}q \quad \text{and} \\ D^2L_0(q) = m|q|^{m-2}(I_d + (m - 2)\bar{q} \otimes \bar{q}) \leq m|q|^{m-2}I_d,$$

where $\bar{q} = q/|q|$. The case $L_0(q) := |q|^m + |q|$ with $m \in (1, 2]$ is also included.

It is clear that L satisfies **(L1)** and **(L2)**. As for **(L3)**, first note that

$$f(x + h) + f(x - h) - 2f(x) \leq |f(x + h) - f(x)| + |f(x - h) - f(x)| \\ \leq 2[f]_\gamma |h|^\gamma.$$

Condition **(L3)** is fulfilled in view of the following result:

Lemma 4.3 *Let $L_0 \in C(\mathbb{R}^d)$ satisfy conditions **(L1)**, **(L2)** and (4.4) for some constants $C_0 > 0$ and $R_0 > 0$. Then, for every $A > 0$, there exists a constant $C_A > 0$ such that*

$$L_0(q + k) + L_0(q - k) - 2L_0(q) \leq C_A|k| \quad \text{for all } (q, k) \in \mathbb{R}^d \times \bar{B}_A. \quad (4.5)$$

Proof Fix $A > 0$. By the fact that the function L_0 is convex and locally bounded, we infer that it is Lipschitz on every ball in \mathbb{R}^d . In particular, there exists a constant $\tilde{C}_A > 0$ such that

$$|L_0(q) - L_0(\eta)| \leq \tilde{C}_A|q - \eta| \quad \text{for all } q, \eta \in \bar{B}_{R_0+2A}.$$

From this, we get

$$L_0(q \pm k) - L_0(q) \leq \tilde{C}_A|k| \quad \text{for all } (q, k) \in \bar{B}_{R_0+A} \times \bar{B}_A.$$

Adding these two yields

$$L_0(q + k) + L_0(q - k) - 2L_0(q) \leq 2\tilde{C}_A|k| \quad \text{for all } (q, k) \in \bar{B}_{R_0+A} \times \bar{B}_A. \quad (4.6)$$

Let $(x_0, q_0) \in \mathbb{T}^d \times (\mathbb{R}^d \setminus \bar{B}_{R_0+A})$. By (4.4) we have

$$D_q^2L_0(q) \leq C_0I_d \quad \text{for all } q \in \bar{B}_A(q_0) \quad (4.7)$$

in the distributional sense. Let $(\rho_\varepsilon)_{\varepsilon>0}$ be a family of smooth mollifiers and set

$$L_\varepsilon(q) := \int_{\mathbb{R}^d} \rho_\varepsilon(\xi)L_0(\xi - q)d\xi \quad \text{for all } q \in \mathbb{R}^d.$$

The function L_ε is smooth and, for $\varepsilon > 0$ small enough, satisfies (4.7) pointwise with the same constant $C_0 > 0$. By the Taylor theorem, for any $k \in \overline{B}_A$ we have

$$\begin{aligned} L_\varepsilon(q_0 \pm k) - L_\varepsilon(q_0) &= \langle D_q L_\varepsilon(q_0), \pm k \rangle + \frac{1}{2} \int_0^1 (1-t) \langle D_q^2 L_\varepsilon(q_0 \pm tk) k, k \rangle dt \\ &\leq \langle D_q L_\varepsilon(q_0), \pm k \rangle + \frac{1}{2} \int_0^1 C_0 |k|^2 dt. \end{aligned}$$

Adding these two yields

$$L_\varepsilon(q_0 + k) + L_\varepsilon(q_0 - k) - 2L_\varepsilon(q_0) \leq C_0 |k|^2 \leq C_0 A |k|.$$

By sending $\varepsilon \rightarrow 0$ we conclude that

$$L_0(q + k) + L_0(q - k) - 2L_0(q) \leq C_0 A |k| \quad \text{for all } (q, k) \in (\mathbb{R}^d \setminus \overline{B}_{R_0+A}) \times \overline{B}_A.$$

This combined with (4.6) implies claim (4.5) with $C_A := \max\{C_0 A, 2\tilde{C}_A\}$.

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