On the relaxation of a class of functionals defined on Riemannian distances

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Abstract

In this paper we study the relaxation of a class of functionals defined on distances induced by isotropic Riemannian metrics on an open subset of $\mathbb{R}^N$. We prove that isotropic Riemannian metrics are dense in Finsler ones and we show that the relaxed functionals admit a specific integral representation.

Keywords: Riemannian and Finsler metrics, relaxation, Gamma convergence

1 Introduction

In this paper we study an integral functional of the form

$$\mathcal{F}(d_a) := \int_\Omega F(x, a(x))\,\mathrm{d}x, \tag{1}$$

defined on the family $\mathcal{I}$ of distances d_a induced by isotropic, continuous Riemannian metrics through the formula

$$d_a(x,y) := \inf\Big\{ \mathrm{L}_a(\gamma) \ : \ \gamma \in \mathrm{Lip}([0,1];\Omega),\ \gamma(0)=x,\ \gamma(1)=y \Big\} \tag{2}$$

for every $(x,y) \in \Omega \times \Omega$, where the length functional L_a is defined as follows

$$\mathrm{L}_a(\gamma) := \int_0^1 a(\gamma(t))|\dot\gamma(t)|\,\mathrm{d}t. \tag{3}$$

Here a varies on the family of positive continuous functions from Ω to the interval $[\alpha, \beta]$, where α and β are fixed positive constants. Distances of this type have already been studied in [6, 3] and, in a more geometric framework, in [8]. The set $\mathcal{I}$ can be seen as a subspace of the space of Finslerian distances $\mathcal{D}$ (see Section 2) endowed with the metrizable topology given by the uniform convergence on compact subset of $\Omega \times \Omega$. It has been proved in [6] that the convergence of a sequence $(d_n)_{n\in\mathbb{N}}$ to d in this topology is equivalent to the Γ-convergence of the associated length functionals L_{d_n} to L_d with respect to the uniform convergence of curves (see Section 2 for definitions). The main problem arising in our study is that $\mathcal{I}$ is not closed with respect to this topology. Indeed, one can build sequences of continuous metrics $(a_n)_{n\in\mathbb{N}}$ which develop an oscillatory behavior in such a way that the induced distances converge to an element d which do not belong to $\mathcal{I}$ (see [1]). Therefore, it is natural to consider the *relaxed functional* of (1), namely

$$\overline{\mathcal{F}}(d) := \inf\{\liminf_n \mathcal{F}(d_n) \ : \ d_n \xrightarrow{\mathcal{D}} d, \ (d_n)_{n\in\mathbb{N}} \subset \mathcal{I}\}, \tag{4}$$

defined for every d belonging to the closure of $\mathcal{I}$, where we have denoted by $\xrightarrow{\mathcal{D}}$ the convergence with respect to the topology of $\mathcal{D}$.

In this paper we prove that the space $\mathcal{I}$ is dense in $\mathcal{D}$ and, under suitable assumptions on the integrand F in (1), that the relaxed functional (4), which is therefore defined on the whole $\mathcal{D}$, has the following integral representation:

$$\overline{\mathcal{F}}(d) = \int_\Omega F(x, \Lambda_d(x))\,\mathrm{d}x, \tag{5}$$

where $\Lambda_d(x) := \sup_{|\xi|=1} \varphi_d(x,\xi)$ and φ_d is the Finslerian metric associated to d by derivation (see Section 2).

We conclude this introduction with some considerations. It is clear by the definition that the relaxed functional $\overline{\mathcal{F}}$ is lower semicontinuous. Moreover, it can be shown that it is the greatest among all lower semicontinuous ones which are bounded from above by $\mathcal{F}$ on $\mathcal{I}$ (see [4] for various results on this topic). Therefore, in order to prove our relaxation result, we have to show first that the functional (5) is lower semicontinuous. The proof of this issue is just a technical adaptation of the arguments described in [5]. To prove the maximality of (5), we will approximate each $d \in \mathcal{D}$ by means of a sequence of suitably chosen distances $d_n \in \mathcal{I}$, namely such that

$$\limsup_n \int_\Omega F(x, \Lambda_{d_n}(x))\,\mathrm{d}x \le \int_\Omega F(x, \Lambda_d(x))\,\mathrm{d}x.$$

Then, by a standard argument (see Section 4), the maximality of (5) follows.
Indeed, finding such an approximating sequence is a delicate matter. In fact, one should define the Riemannian metrics a_n in such a way to have Γ-convergence of the relative length functionals L_{a_n} to L_{φ_d} and this problem is not trivial even in the simplified situation of an isotropic Riemannian metric φ_d, i.e. such that $\varphi_d = b(x)|\xi|$ where b is a Borel function from Ω to $[\alpha, \beta]$. It is clear, in fact, that this convergence strongly relies upon the convergence of the approximating metrics on curves, which is much finer than convergence almost everywhere in Ω. Moreover we do not have much information on the properties of the metric φ_d; we only know it is Borel measurable and such that the associated length functional L_{φ_d} is lower semicontinuous with respect to the uniform convergence of curves (see Section 2). In the general case of a non-isotropic metric the situation is obviously more delicate.
The key idea of our proof is that it is sufficient to control the convergence of the approximating distances only on a fixed countable and dense subset of $\Omega \times \Omega$ (Lemma 3.7). Therefore, when we define the Riemannian metrics, we have only to control the value of the associated distance d_n on the first n points of the countable, dense subset. This will be done by approximating the Finsler metric φ_d along geodesics (or, more precisely, quasi-geodesics, see (20)).
The problem of the density of (smooth) isotropic, Riemannian metrics in Finsler ones has already been studied. The question was raised in [6], and partially answered in [3] under the additional assumption that φ_d is lower semicontinuous in the first variable. We remark that our proof does not require any assumption on the Finsler metric and therefore completely answers to the question. Indeed, as pointed out in [3], once we have the density result for continuous and isotropic Riemannian metrics, the analogous result for smooth ones is easily recovered via a regularization argument (see Remark 4.4).

We conclude the paper by showing that every Finsler distance $d \in \mathcal{D}$ can indeed be seen as generated by a suitable Borel measurable, isotropic Riemannian metric $a : \overline{\Omega} \to [\alpha, \beta]$ (according to definition (28), see Proposition 4.8). In other words, by allowing the isotropic metric a to vary in a somehow "uncontrolled" way, one can recover all the possible anisotropies of φ_d.

The paper is organized as follows: in Section 2 we recall the main notation used in the sequel and some results on Finsler metrics, Section 3 contains some preliminary lemmas and in Section 4 we prove our main results.

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2 Notation and preliminaries on Finsler metrics

We write here a list of symbols used throughout this paper.

Ω	an open subset of $\mathbb{R}^N$
$\mathbb{S}^{N-1}$	the unitary sphere of $\mathbb{R}^N$
$B_r(x)$	the open ball in $\mathbb{R}^N$ of radius r centred in x
I	the closed interval $[0, 1]$
$\mathcal{L}^N$	the $N-$dimensional Lebesgue measure
$\mathcal{H}^N$	the $N-$dimensional Hausdorff measure
$\|u\|$	the Euclidean norm of the vector $u \in \mathbb{R}^N$
χ_E	the characteristic function of the set E
$\text{argmin}(\mathcal{P})$	the set of minimizers of the problem $(\mathcal{P})$

In this paper the letter N denotes an integer number greater or equal to 2. We will say that a set ω is *well contained* in Ω and we will write $\omega \subset\subset \Omega$ to mean that its closure $\overline{\omega}$ is contained in Ω. With the word *curve* or *path* we will always indicate a Lipschitz function from the interval $I := [0, 1]$ to an open subset Ω of $\mathbb{R}^N$. Any curve γ is always supposed to be parametrized by constant speed, i.e. in such a way that $|\dot{\gamma}(t)|$ is constant for $\mathcal{L}^1$-a.e. $t \in I$. We will say that a sequence of curves $(\gamma_n)_{n\in\mathbb{N}}$ (uniformly) converges to a curve γ to mean that $\sup_{t\in I} |\gamma_n(t) - \gamma(t)|$ tends to zero as n goes to infinity. We will denote by $\mathcal{L}_{x,y}$ the family of curves γ which join x to y, i.e. such that $\gamma(0) = x$ and $\gamma(1) = y$. We remark that if a sequence of curves $(\gamma_n)_{n\in\mathbb{N}} \subset \mathcal{L}_{x,y}$ is such that $\sup_n \int_0^1 |\dot{\gamma}(t)| \mathrm{d}t < +\infty$ then, since they are all parametrized by constant speed, we have that their first derivative is bounded from above. Therefore, by applying Ascoli-Arzelà theorem, we can find a curve $\gamma \in \mathcal{L}_{x,y}$ such that a subsequence $(\gamma_{n_i})_{i\in\mathbb{N}}$ converges to γ. This argument will be widely used throughout the paper with no further explanation.

The function $F : \Omega \times [\alpha, \beta] \to \mathbb{R}$ appearing in the integrand of (1) is assumed to be continuous and to fulfill the following conditions:

$$\begin{aligned} &\text{(i) the function } F(x, \cdot) \text{ is convex and nondecreasing for } \mathcal{L}^N\text{-a.e. } x \in \Omega; \\ &\text{(ii) } \int_\Omega F(x, \beta)\, \mathrm{d}x < +\infty. \end{aligned} \tag{6}$$

We recall the notion of Γ-convergence. Let (X, τ) be a topological space satisfying the first axiom of countability at the point $x \in X$. A sequence of functionals $F_n : X \to \overline{\mathbb{R}}$ is said to Γ-converge at x if

$$\Gamma - \liminf F_n(x) = \Gamma - \limsup F_n(x),$$

where

$$\begin{cases} \Gamma - \liminf F_n(x) := \inf \{ \liminf_n \ F_n(x_n) \ : \ x_n \stackrel{\tau}{\longrightarrow} x \ \} \\ \Gamma - \limsup F_n(x) := \inf \{ \limsup_n F_n(x_n) \ : \ x_n \stackrel{\tau}{\longrightarrow} x \ \}. \end{cases}$$

Definition 2.1. A Borel function $\varphi : \Omega \times \mathbb{R}^N \to [0, +\infty)$ is said to be a *Finsler metric* on the open set $\Omega \subset \mathbb{R}^N$ if the function $\varphi(x, \cdot)$ is positively 1-homogeneous for every $x \in \Omega$ and convex for $\mathcal{L}^N$-a.e. $x \in \Omega$.

Given a Finsler metric, we can define a distance d_φ on Ω through the formula

$$d_\varphi(x, y) := \inf \{ \mathrm{L}_\varphi(\gamma) \mid \gamma \in \mathcal{L}_{x,y} \}, \tag{7}$$

where the Finslerian length functional L_φ is defined by

$$\mathrm{L}_\varphi(\gamma) := \int_0^1 \varphi(\gamma(t), \dot{\gamma}(t)) \mathrm{d}t.$$

A distance deriving from a Finsler metric through (7) is said to be of *Finsler type.* We will say that a distance d is *locally equivalent* to the Euclidean one if, for every $x \in \Omega$, there exists an open neighborhood U_x and some positive constants c_x, C_x such that $c_x|x - y| \leq d(x, y) \leq C_x|x - y|$ for every $y \in U_x$. We will say that a distance function is of *geodesic type* if it satisfies the following identity:

$$d(x, y) = \inf \left\{ \mathrm{L}_d(\gamma) \mid \gamma \in \mathcal{L}_{x,y} \right\} \qquad \text{for every } (x, y) \in \Omega \times \Omega, \tag{8}$$

where $\mathrm{L}_d(\gamma)$ denotes the classical d-length of γ, obtained as the supremum of the d-lengths of inscribed polygonal curves:

$$\mathrm{L}_d(\gamma) := \sup \left\{ \sum_i d\big(\gamma(t_i), \gamma(t_{i+1})\big) \ : \ 0 = t_0 < t_1 < .. < t_r = 1, \ r \in \mathbb{N} \right\} . \tag{9}$$

It can be easily shown by the definition that

Proposition 2.2. *The length functional* L_d *is lower semicontinuous with respect to the uniform convergence of paths, namely if* $(\gamma_n)_{n\in\mathbb{N}}$ *converges to* γ *then*

$$\mathrm{L}_d(\gamma) \leq \liminf_{n\to+\infty} \mathrm{L}_d(\gamma_n).$$

If the distance d is locally equivalent to the Euclidean one, then it can be proved (cf. [8]) that the length functional L_d admits the integral representation

$$\mathrm{L}_d(\gamma) = \int_0^1 \varphi_d(\gamma(t), \dot{\gamma}(t)) \, \mathrm{d}t$$

for every curve γ, where φ_d is the Finsler metric associated to d by derivation, namely

$$\varphi_d(x, \xi) := \limsup_{t\to 0^+} \frac{d(x, x + t\xi)}{t} \qquad (x, \xi) \in \Omega \times \mathbb{R}^N .$$

Denote by $d_\Omega(x, y)$ the Euclidean geodesic distance in Ω, that is $d_\Omega := d_a$ according to (2), with a identically equal to 1. We remark that d_Ω locally coincides with the Euclidean distance. We fix two positive constants α, β with $\beta > \alpha$ and we set

$$\mathcal{M} := \{ \varphi \text{ Finsler metric on } \Omega \ : \ \alpha |\xi| \leq \varphi(x, \xi) \leq \beta |\xi| \}.$$

Then we define the family $\mathcal{D}$ of distances on Ω generated by the metrics $\mathcal{M}$, namely $\mathcal{D} := \{ d_\varphi \mid \varphi \in \mathcal{M} \}$. Obviously the set $\mathcal{I}$, made up by distances d_a defined by (2) with $a : \Omega \to [\alpha, \beta]$ continuous, is trivially included in $\mathcal{D}$ identifying $a(x)$ with the metric $a(x)|\xi|$. It is also evident that $\alpha \, d_\Omega \leq d \leq \beta \, d_\Omega$ for every $d \in \mathcal{D}$, so such distances are locally equivalent to the Euclidean one. Moreover one can easily show the following result.

Proposition 2.3. *Let $d := d_\varphi$ for some $\varphi \in \mathcal{M}$. Then $\mathrm{L}_d(\gamma) \leq \mathrm{L}_\varphi(\gamma)$ for every curve γ. In particular, d is a distance of geodesic type according to definition (8).*

Remark 2.4. The inequality in the previous proposition may be strict. For example, take $\Omega := [-1,1]\times[-1,1]$, $\Gamma := \{0\}\times[-1,1]$ and $a(x) := \chi_\Omega(x)+\chi_\Gamma(x)$. Then $d_a(x,y) = |x-y|$. If now we take $\gamma(t) := (0,-1/2)(1-t)+(0,1/2)t$, it is easily seen that $\mathrm{L}_{d_a}(\gamma) = 1 < 2 = \mathrm{L}_a(\gamma)$.

We endow $\mathcal{D}$ with the topology given by the uniform convergence on compact subset of $\Omega \times \Omega$. We will write $d_n \xrightarrow{\mathcal{D}} d$ to mean that the sequence $(d_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ converges to $d \in \mathcal{D}$ with respect to this topology. It has been proved [6, Theorem 3.1] that this convergence is equivalent to the Γ-convergence of the relative length functionals with respect to the uniform convergence of paths. Moreover, we have the following result (compare to [5, Proposition 4] and [6, Theorem 3.1]):

Proposition 2.5. *Let Ω be an open subset of $\mathbb{R}^N$ such that $d_\Omega(x,y) \leq C_r|x-y|$ for every x and y in $\Omega \cap B_r(0)$ and every $r > 0$, where C_r is some positive constant which depends on r. Then $\mathcal{D}$ is a metrizable compact space.*

Throughout this paper we will always work with sets Ω which satisfy the condition stated in the proposition above. Therefore we will always assume that $\mathcal{D}$ is compact. In particular, this holds whenever Ω has a locally Lipschitz boundary.

Given a distance $d \in \mathcal{D}$, we define for every $x \in \Omega$

$$\Lambda_d(x) := \sup_{|\xi|=1} \varphi_d(x,\xi), \tag{10}$$

which represents, with analogy to the Riemannian case $\varphi_d(x,\xi) = B(x)\xi \cdot \xi$ with $B(x)$ a symmetric and positive definite matrix, the largest "eigenvalue" of $\varphi_d(x,\cdot)$ at the point x. We notice that $\Lambda_d(x)$ is a Lebesgue measurable function. Indeed, if $(\xi_n)_{n\in\mathbb{N}}$ is a dense sequence in $\mathbb{S}^{N-1}$, we have that $\Lambda_d(x)$ coincides with the Borel measurable function $\sup_n \varphi_d(x,\xi_n)$ on $\Omega \setminus E$, where E is the set of points where $\varphi_d(x,\cdot)$ is not continuous. We know that $\varphi_d(x,\cdot)$ is convex for almost every x by definition of Finsler metric, therefore E is $\mathcal{L}^N$-negligible and the claim follows.

3 Preliminary results

In this section we prepare the tools which will be used in the proof of our relaxation results.

We recall that the function $F : \Omega \times [\alpha,\beta] \to \mathbb{R}$ is continuous and fulfills conditions (6). We have

Lemma 3.1. *Let $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ such that $d_{\varphi_n} \xrightarrow{\mathcal{D}} d$ for some $d \in \mathcal{D}$. Then, for every bounded Borel set $\omega \subset\subset \Omega$ and every $\xi \in \mathbb{S}^{N-1}$, we have*

$$\int_\omega F\left(x,\varphi_d(x,\xi)\right) \mathrm{d}x \leq \liminf_{n\to\infty} \int_\omega F\left(x,\varphi_n(x,\xi)\right) \mathrm{d}x\ .$$

Proof : Let ω be a bounded Borel set well contained in Ω. Choose a bounded open set $A \subset\subset \Omega$ that contains ω. Arguing as in the proof of [5, Proposition 9], for every fixed $\xi \in \mathbb{S}^{N-1}$ it is possible to find a subsequence of $(\varphi_n)_{n\in\mathbb{N}}$ and a sequence of positive numbers $t_n \to 0$ such that, for a.e. $x \in A$,

$$F(x,\varphi\left(x,\xi\right)) = \lim_{n\to\infty} \chi_{A_n}(x) F\left(x, \frac{d_{\varphi_n}\left(x,x+t_n\xi\right)}{t_n}\right), \tag{11}$$

where $A_n := \{x \in A \,|\, \mathrm{dist}(x, \partial A) > t_n\}$. Now, integrating (11) over ω and applying the dominated convergence theorem, we get:

$$\int_\omega F(x, \varphi(x,\xi))\,\mathrm{d}x = \lim_{n\to\infty} \int_\omega \chi_{A_n}(x) F\left(x, \frac{d_{\varphi_n}(x, x+t_n\xi)}{t_n}\right) \mathrm{d}x. \tag{12}$$

Since $d_{\varphi_n}(x, x+t_n\xi)$ is less than or equal to the (Finslerian) length of the straight line segment joining x and $x+t_n\xi$, we have

$$d_{\varphi_n}(x, x+t_n\xi) \leq \int_0^1 \varphi_n(x+st_n\xi, t_n\xi)\,\mathrm{d}s.$$

By the monotonicity and convexity of the function $F(x,\cdot)$ for a.e. x we get, by using Jensen inequality, that for a.e. $x \in A$,

$$F\left(x, \frac{d_{\varphi_n}(x, x+t_n\xi)}{t_n}\right) \leq \int_0^1 F(x, \varphi_n(x+st_n\xi, \xi))\mathrm{d}s. \tag{13}$$

Combining (12) and (13), we obtain

$$\begin{aligned}
\int_\omega F(x, \varphi(x,\xi))\mathrm{d}x &\leq \liminf_{n\to\infty} \int_\Omega \chi_{A_n\cap\omega}(x) \int_0^1 F(x, \varphi_n(x+st_n\xi, \xi))\mathrm{d}s\,\mathrm{d}x \\
&= \liminf_{n\to\infty} \int_0^1 \int_\Omega \chi_{A_n\cap\omega}(x-st_n\xi)\, F(x-st_n\xi, \varphi_n(x,\xi))\mathrm{d}x\,\mathrm{d}s \\
&= \liminf_{n\to\infty} \int_\omega F(x, \varphi_n(x,\xi))\mathrm{d}x\ .
\end{aligned}$$

□

The following two lemmas are analogous to [5, Lemma10, Lemma 11] and may be proved in the same way, up to some technical adaptations.

Lemma 3.2. *Let $\varphi \in \mathcal{M}$ be a continuous Finsler metric. Then, for every bounded open set $A \subset\subset \Omega$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\int_{D_i^\delta\cap A} F(x, \Lambda_\varphi(x))\,\mathrm{d}x \leq \sup_{|\xi|=1} \int_{D_i^\delta\cap A} \left[F(x, \varphi(x,\xi)) + \varepsilon\right] dx \qquad \textit{for all } i \in \mathbb{Z}^N,$$

where we have set $D_i^\delta := \Omega \cap \left(i + [-\delta,\delta)^N\right)$.

Lemma 3.3. *Let $\varphi \in \mathcal{M}$ such that $\varphi(x,\cdot)$ is convex for every $x \in \Omega$. Then for every bounded open set $A \subset\subset \Omega$ and for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset A$ such that $\mathcal{L}^N(A \setminus K_\varepsilon) < \varepsilon$ and φ is continuous on $K_\varepsilon \times \mathbb{R}^N$.*

By using the previous lemmas we can prove the following

Proposition 3.4. *Let $\varphi \in \mathcal{M}$. Assume that, for a sequence $(\mu_n)_{n\in\mathbb{N}}$ of nonnegative Borel measures on Ω, the following property holds:*

$$\sup_{|\xi|=1} \int_\omega F(x, \varphi(x,\xi))\,\mathrm{d}x \leq \liminf_{n\to\infty} \mu_n(\omega) \qquad \textit{for every Borel set } \omega \subset\subset \Omega\ .$$

Then

$$\int_\Omega F(x, \Lambda_\varphi(x))\,\mathrm{d}x \leq \liminf_{n\to\infty} \mu_n(\Omega)\,. \tag{14}$$

Proof : Let $(\Omega_l)_{l\in\mathbb{N}}$ be a sequence of bounded open sets well contained in Ω such that $\overline{\Omega}_l \subset \Omega_{l+1}$ and $\Omega = \bigcup_{l\in\mathbb{N}} \Omega_l$. We first remark that it is sufficient to prove that (14) holds for $\Omega := \Omega_l$ for every $l \in \mathbb{N}$. Then, the claim is easily obtained by adapting the proof given in [5, Proposition 12] and by using Lemmas 3.1, 3.2 and 3.3. □

Next, we show some results on Finsler metrics. We start by the following

Proposition 3.5. *Let $\varphi \in \mathcal{M}$ and $d := d_\varphi$. Then*

(i) $\varphi_d(x,\xi) \leq \varphi(x,\xi)$ for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$. In particular $\Lambda_d(x) \leq \sup_{|\xi|=1} \varphi(x,\xi)$ for a.e. $x \in \Omega$;

(ii) if $\varphi(x,\xi) := a(x)|\xi|$ with $a : \Omega \to [\alpha,\beta]$ lower semicontinuous, then $\varphi_d(x,\xi) \geq a(x)|\xi|$ for every $(x,\xi) \in \Omega \times \mathbb{R}^N$. In particular $a(x) = \Lambda_d(x)$ for a.e. $x \in \Omega$.

Proof : Let us fix a $\xi \in \mathbb{S}^{N-1}$. For every $x \in \Omega$ let us define the curve $\gamma_x(t) := x + t\xi$. Then by Proposition 2.3 we have that

$$\mathrm{L}_d(\gamma_x) := \int_0^1 \varphi_d(\gamma_x,\xi)\,\mathrm{d}t \leq \int_0^1 \varphi(\gamma_x,\xi)\,\mathrm{d}t =: \mathrm{L}_\varphi(\gamma_x).$$

Therefore we deduce that $\varphi_d(x,\xi) \leq \varphi(x,\xi)$ for a.e. $x \in \Omega$. Then we can take a dense sequence $(\xi_n)_{n\in\mathbb{N}}$ in $\mathbb{S}^{N-1}$ and repeat the argument above for each ξ_n. Recalling that the functions $\varphi_d(x,\cdot)$ and $\varphi(x,\cdot)$ are continuous and 1-homogeneous for a.e. $x \in \Omega$, we eventually get, by the density of $(\xi_n)_{n\in\mathbb{N}}$, that $\varphi_d(x,\xi) \leq \varphi(x,\xi)$ for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$. In particular we get

$$\Lambda_d(x) \leq \sup_{|\xi|=1} \varphi(x,\xi) \qquad \text{a.e. in } \Omega. \tag{15}$$

Let us now take $\varphi(x,\xi) := a(x)|\xi|$ with a lower semicontinuous. Then we have, by the lower semicontinuity, that $a(x) = \sup_{r>0} \left(\inf_{B_r(x)} a\right)$. Therefore for every fixed $x \in \Omega$ and for every $\varepsilon > 0$ there exists $r_\varepsilon > 0$ such that $B_{r_\varepsilon}(x) \subset \Omega$ and $a(y) \geq a(x) - \varepsilon$ for every $y \in B_{r_\varepsilon}(x)$. Let us fix a $\xi \in \mathbb{S}^{N-1}$ and take $0 < t < \alpha r_\varepsilon/(2\beta)$. Choose a d-minimizing sequence $(\gamma_n)_{n\in\mathbb{N}} \subset \mathcal{L}_{x,x+t\xi}$ such that $\mathrm{L}_a(\gamma_n) \leq d(x,x+t\xi) + \alpha r_\varepsilon/2$ for every n. Then the curves γ_n lie within $B_{r_\varepsilon}(x)$. In fact for every n and for every $s \leq 1$:

$$\alpha|\gamma_n(s) - x| \leq \int_0^s a(\gamma)|\dot\gamma_n|\mathrm{d}\tau \leq d(x,x+t\xi) + \alpha r_\varepsilon/2 \leq \beta d_\Omega(x,x+t\xi) + \alpha r_\varepsilon/2 < \alpha r_\varepsilon,$$

where we have used the fact that $d_\Omega(x,y) = |x-y|$ if $y \in B_{r_\varepsilon}(x)$. Then we have for every n

$$\mathrm{L}_a(\gamma_n) := \int_0^1 a(\gamma_n)|\dot\gamma_n|\,\mathrm{d}\tau \geq (a(x)-\varepsilon)\int_0^1 |\dot\gamma_n|\,\mathrm{d}\tau \geq (a(x)-\varepsilon)t$$

and letting n go to infinity we obtain

$$\frac{d(x,x+t\xi)}{t} \geq a(x) - \varepsilon. \tag{16}$$

By passing to the limsup in (16) as $t \to 0$ and since $\varepsilon > 0$, $x \in \Omega$ and $\xi \in \mathbb{S}^{N-1}$ were arbitrary we obtain

$$\varphi_d(x,\xi) \geq a(x) \qquad \text{for every } (x,\xi) \in \Omega \times \mathbb{S}^{N-1} \tag{17}$$

and the claim follows by the 1-homogeneity of $\varphi_d(x,\cdot)$. In particular, by taking the sup of the left-hand side of (17) over all $\xi \in \mathbb{S}^{N-1}$ and by using (15) we get that $\Lambda_d(x) = a(x)$ for a.e. $x \in \Omega$. □

Remark 3.6. If a and b are two continuous isotropic metrics which give rise to the same distance function d through (2), then $a(x) = b(x)$ for every x in Ω. In fact, by point (ii) of the stated lemma, we have that the previous equality holds almost everywhere, and therefore everywhere by the continuity of the metrics. In particular, this shows that the functional (1) is well definite.

The key idea used in the proof of the density result is stated in the following

Lemma 3.7. *Let $(d_n)_{n\in\mathbb{N}}$ be a sequence contained in $\mathcal{D}$ which converges pointwise to some $d \in \mathcal{D}$ on a dense subset of $\Omega \times \Omega$. Then $d_n \xrightarrow{\mathcal{D}} d$.*

Proof : By the compactness of $\mathcal{D}$, we already know that there is a subsequence $(d_{n_k})_{k\in\mathbb{N}}$ such that $d_{n_k} \xrightarrow{\mathcal{D}} \delta$ for some $\delta \in \mathcal{D}$. By the pointwise convergence we get that $\delta(x, y) = d(x, y)$ on a dense subset of $\Omega \times \Omega$ and therefore δ coincides with d since they are both continuous functions. If the whole sequence did not converge uniformly (on compact subset of $\Omega \times \Omega$) to d, by the compactness of $\mathcal{D}$ there would exists a subsequence which converges to some $\delta \in \mathcal{D}$ with $\delta \neq d$. By arguing as above, this would lead to a contradiction. □

The next result shows that the monotone convergence of metrics implies the convergence of the induced distances.

Lemma 3.8. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}$ such that for every $(x,\xi) \in \Omega \times \mathbb{R}^N$ $\varphi_n(x,\xi)$ converge increasingly (resp. decreasingly) to $\varphi(x,\xi)$ for some $\varphi \in \mathcal{M}$. Then $d_{\varphi_n} \xrightarrow{\mathcal{D}} d_\varphi$.*

Proof : By Lemma 3.7 it is sufficient to prove that $(d_{\varphi_n})_{n\in\mathbb{N}}$ converges pointwise to d_φ. We start by considering the case of an increasing sequence of metrics. By the monotonicity of φ_n we obviously have that $\mathrm{L}_\varphi(\gamma) \geq \mathrm{L}_{\varphi_n}(\gamma) \geq \mathrm{L}_{\varphi_{n-1}}(\gamma)$ for every curve γ and therefore $(d_{\varphi_n}(x,y))_{n\in\mathbb{N}}$ is an increasing sequence and $d_\varphi(x,y) \geq \sup_n d_{\varphi_n}(x,y)$ for every $(x,y) \in \Omega \times \Omega$. To prove the reverse inequality, let us take a sequence of curves $(\gamma_n)_{n\in\mathbb{N}} \subset \mathcal{L}_{x,y}$ such that $L_{\varphi_n}(\gamma_n) \leq d_{\varphi_n}(x,y) + 1/n$. Since the functionals L_{φ_n} are equi-coercive (in fact $L_{\varphi_n}(\gamma) \geq \alpha \int_0^1 |\dot{\gamma}| dt$ for every n), we may find a subsequence $(\gamma_{n_i})_{i\in\mathbb{N}}$ which converges uniformly to some curve $\gamma \in \mathcal{L}_{x,y}$. Now, by [7, Remark 5.5], we know that the functionals $\mathrm{L}_{\varphi_{n_i}}$ Γ-converge to L_φ with respect to the uniform convergence of path and therefore we have

$$d_\varphi(x,y) \leq \mathrm{L}_\varphi(\gamma) \leq \liminf_{i\to+\infty} \mathrm{L}_{\varphi_{n_i}}(\gamma_{n_i}) \leq \liminf_{i\to+\infty} d_{\varphi_{n_i}}(x,y) = \sup_n d_{\varphi_n}(x,y).$$

Since $(x,y) \in \Omega \times \Omega$ was arbitrary the claim follows.
The proof in the case of a decreasing sequence of metrics is even simpler. In fact, by monotonicity we get $d_\varphi(x,y) \leq \inf_n d_{\varphi_n}(x,y)$ for every $(x,y) \in \Omega \times \Omega$. To show the reverse inequality, take a curve $\gamma \in \mathcal{L}_{x,y}$. By the monotone convergence theorem and by the definition of $d_{\varphi_n}(x,y)$ we have

$$\mathrm{L}_\varphi(\gamma) = \inf_n \mathrm{L}_{\varphi_n}(\gamma) \geq \inf_n d_{\varphi_n}(x,y),$$

and the claim easily follows by taking the infimum over all curves in $\mathcal{L}_{x,y}$. □

We end this section with the proof of two lemmas which will be useful in the sequel.

Lemma 3.9. *Let $\{\gamma^i \mid \gamma^i \in \mathcal{L}_{x_i,y_i},\, i \leq n\}$ be a finite collection of curves such that*

$$d(x_i,y_i) \leq L_d(\gamma^i) \leq d(x_i,y_i) + \frac{1}{n} \tag{18}$$

for some fixed points $(x_i,y_i) \in \Omega \times \Omega$ and for some $n \in \mathbb{N}$. Then it is possible to find a family of curves $\{\tilde{\gamma}^i \mid \tilde{\gamma}^i \in \mathcal{L}_{x_i,y_i},\, i \leq n\}$ still satisfying (18) and such that *(i)* $\tilde{\gamma}^i$ *is injective for every* $i \leq n$*;*

(ii) $\tilde{\gamma}^i(I) \cap \tilde{\gamma}^j(I)$ *is a (possibly void) disjoint finite union of closed arcs for every* $1 \leq i \leq j \leq n$*.*

Proof : Let N be a 1-rectifiable closed set such that $N \supset \cup_{i \leq n} \gamma^i$. First we remark that for every $i \leq n$ the set

$$\mathcal{R}_i := \operatorname{argmin}\{\mathrm{L}_d(\gamma) \,|\, \gamma \in \mathcal{L}_{x_i,y_i}, \gamma(I) \subset N\}$$

is non-void. Indeed, the class of curves on which we minimize L_d is non-void, as it contains γ^i, and closed with respect to the uniform convergence of curves, as N is closed, therefore it contains an accumulation point $\tilde{\gamma}^i$ of a minimizing sequence. Such a curve is of minimal d-length by the lower semicontinuity of L_d and so it belongs to $\mathcal{R}_i$. Moreover, it is injective and satisfies (18) by minimality.
The proof of the lemma is by induction on n. For $n = 1$ the claim is satisfied by choosing a $\tilde{\gamma}^1$ which belongs to $\mathcal{R}_1$. Let us then suppose the claim satisfied up to $n-1$ and let us prove it for n. By induction we may find curves $\tilde{\gamma}^i \in \mathcal{R}_i$ for $i \leq n-1$ such to satisfy the claim. Let us choose a curve σ in $\mathcal{R}_n$. For every $j \leq n-1$ let us set $t_j := \min\{t \in I \,|\, \sigma(t) \in \tilde{\gamma}^j(I)\}$ and $T_j := \max\{t \in I \,|\, \sigma(t) \in \tilde{\gamma}^j(I)\}$. Up to reordering the curves $\tilde{\gamma}^j$, we can suppose that $t_1 = \min\{t_j \,|\, j \leq n-1\}$. Then we define $\tau_1 \in \mathcal{L}_{x_n,y_n}$ to be the curve obtained by moving from $\sigma(0)$ to $\sigma(t_1)$ along σ, from $\sigma(t_1)$ to $\sigma(T_1)$ along $\tilde{\gamma}_1$ and from $\sigma(T_1)$ to $\sigma(1)$ along σ again. Remark that, by minimality, $\tilde{\gamma}^1$ is a path which connects $\sigma(t_1)$ to $\sigma(T_1)$ in the shortest way among all those contained in N and so we have not increased the length, i.e. $\mathrm{L}_d(\tau_1) \leq \mathrm{L}_d(\sigma)$ and $\tau_1 \in \mathcal{R}_n$. Moreover $\tau_1([0,T_1]) \cap \tilde{\gamma}_i(I)$ is a disjoint finite union of closed arcs for every $1 \leq i \leq n-1$. Then we set $\sigma := \tau_1 \,|_{[T_1,1]}$ and we repeat the argument above to obtain a $\tau_2 : [T_1, 1] \to N$. By iterating this procedure we eventually find a finite number of curves $\{\tau_h \,|\, 1 \leq h \leq M\}$ for some $M < n$. Then we define

$$\tilde{\gamma}^n(t) := \begin{cases} \tau_1(t) & \text{if } t \in [0, T_1] \\ \tau_h(t) & \text{if } t \in [T_{h-1}, T_h] \text{ and } 1 < h < M \\ \tau_M(t) & \text{if } t \in [T_{M-1}, 1]. \end{cases}$$

By what previously observed, we have that $\tilde{\gamma}^n$ still belongs to $\mathcal{R}_n$ and is therefore injective by minimality. Moreover, it is such that $\tilde{\gamma}^n(I) \cap \tilde{\gamma}^i(I)$ is a disjoint finite union of closed arcs for every $i \leq n-1$ by construction. The claim is thus proved. □

Lemma 3.10. *Let* γ *be an injective curve,* $\Gamma := \gamma((0,1)) \subset \Omega$ *and* $a : \Omega \to [\alpha, \beta]$ *a Borel function. Then there exists a sequence of continuous functions* $\sigma_k : \Gamma \to [\alpha, \beta]$ *such that* $\sigma_k(x)$ *converge to* $a(x)$ *for* $\mathcal{H}^1$*-a.e.* $x \in \Gamma$*. Moreover, for every* $\varepsilon > 0$ *there exists a Borel subset* $B_\varepsilon \subset \Gamma$ *such that* $\mathcal{H}^1(\Gamma \setminus B_\varepsilon) < \varepsilon$ *and* σ_k *converge uniformly to* a *on* B_ε*.*

Proof : The function $a \circ \gamma : (0,1) \to [\alpha, \beta]$ is Borel measurable, therefore there exists a sequence $(f_k)_{k \in \mathbb{N}}$ of continuous functions $f_k : (0,1) \to [\alpha, \beta]$ such that $f_k(t)$ converges to $a \circ \gamma(t)$ for a.e. $t \in (0,1)$. Moreover, by Severini-Egoroff's theorem [9, Section 1.2, Theorem 3], for every $\varepsilon > 0$ there exist an infinitesimal sequence $(\delta_k)_{k \in \mathbb{N}}$ and a Borel set E_ε such that $\mathcal{H}^1((0,1) \setminus E_\varepsilon) < \varepsilon$ and $|f_k(t) - a \circ \gamma(t)| < \delta_k$ for every $t \in E_\varepsilon$. The claim then follows by setting $\sigma_k(x) := f_k(\gamma^{-1}(x))$ and $B_\varepsilon := \gamma(E_\varepsilon)$. □

4 Main results

Our main result is stated as follows.

Theorem 4.1. *Let $\mathcal{F}$ be the functional defined on $\mathcal{I}$ by (1), where $F : \Omega \times [\alpha, \beta] \to \mathbb{R}$ is continuous and satisfies conditions (6). Then its relaxed functional (4) has the following integral representation:*

$$\overline{\mathcal{F}}(d) = \int_\Omega F(x, \Lambda_d(x))\,\mathrm{d}x \tag{19}$$

for all $d \in \mathcal{D}$.

The proof of the theorem above is based on the following two results which we state separately.

Theorem 4.2. *If $d_n \xrightarrow{\mathcal{D}} d$, then $\liminf_{n\to+\infty} \int_\Omega F(x, \Lambda_{d_n}(x))\,\mathrm{d}x \geq \int_\Omega F(x, \Lambda_d(x))\,\mathrm{d}x$.*

Theorem 4.3. *The family $\mathcal{I}$ of distances induced by continuous and isotropic Riemannian metrics is dense in $\mathcal{D}$. Moreover, for every $d \in \mathcal{D}$ we can choose a sequence $(d_n)_{n\in\mathbb{N}} \subset \mathcal{I}$ such that $d_n \xrightarrow{\mathcal{D}} d$ and*

$$\limsup_{n\to+\infty} \int_\Omega F(x, \Lambda_{d_n}(x))\,\mathrm{d}x \leq \int_\Omega F(x, \Lambda_d(x))\,\mathrm{d}x.$$

Remark 4.4. The class of distances induced by smooth isotropic Riemannian metrics is dense in $\mathcal{I}$. Therefore, by the theorem just stated, smooth isotropic Riemannian metrics are dense in the class of Finsler metrics. In fact, let us take a distance d in $\mathcal{I}$. Then $d = d_a$ for some continuous metric $a : \Omega \to [\alpha, \beta]$. We may extend a to the whole $\mathbb{R}^n$ by setting a identically equal to α outside Ω. Then, by taking a sequence of convolution kernels ρ_n, we define the sequence of smooth isotropic metrics $a_n : \Omega \to [\alpha, \beta]$ by regularization, i.e. $a_n(x) := \rho_n * a(x)$, and we call d_n the induced distances. Since the functions a_n converge to a uniformly on compact subset of $\Omega \times \Omega$, it can be easily shown that the length functionals L_{a_n} Γ-converge to L_a with respect to the uniform convergence of curves. Then, by [6, Theorem 3.1], we have that $d_n \xrightarrow{\mathcal{D}} d$ (this could also have been proved directly by using the equi-coercivity of the length functionals to show that the above convergence of distances is pointwise and then applying Lemma 3.7).

Once Theorem 4.2 and Theorem 4.3 are proved, the proof of Theorem 4.1 will trivially follows. In fact, Theorem 4.2 gives that the functional (19) is lower semicontinuous with respect to the uniform convergence of distances, and Theorem 4.3 implies it is the greatest lower semicontinuous functional defined on $\mathcal{D}$ which is bounded from above by $\mathcal{F}$ on $\mathcal{I}$. In fact, let $\mathcal{G}$ be another candidate and let $d \in \mathcal{D}$. Choose a sequence $(d_n)_{n\in\mathbb{N}} \subset \mathcal{I}$ as in the statement of Theorem 4.3. We have

$$\mathcal{G}(d) \leq \liminf_{n\to+\infty} \mathcal{G}(d_n) \leq \liminf_{n\to+\infty} \mathcal{F}(d_n) \leq \limsup_{n\to+\infty} \mathcal{F}(d_n) \leq \int_\Omega F(x, \Lambda_d(x))\,\mathrm{d}x,$$

hence the claim. We remark that by Proposition 3.5 the functional (19) actually coincides with $\mathcal{F}$ on $\mathcal{I}$.

Let us then start by proving Theorem 4.2.

Proof of Theorem 4.2: By applying Lemma 3.1 with $\varphi_n := \varphi_{d_n}$, we obtain

$$\sup_{|\xi|=1} \int_\omega F(x, \varphi_d(x,\xi))\,\mathrm{d}x \leq \liminf_{n\to\infty} \int_\omega F(x, \Lambda_{d_n}(x))\,\mathrm{d}x.$$

The claim then follows by applying Proposition 3.4 with $\mu_n(\omega) := \int_\omega F(x, \Lambda_{d_n}(x))\,\mathrm{d}x$. □

Remark 4.5. The proof above still works for slightly more general functionals. Indeed, it is sufficient that there exists a sequence of continuous functions $F_k : \Omega \times [\alpha, \beta] \to \mathbb{R}$ which satisfy conditions (6) and such that $F(x, \xi) = \sup_k F_k(x, \xi)$ for $\mathcal{L}^N$-a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$. In fact, one can apply the argument above to each F_k to get

$$\int_\omega F_k\left(x, \Lambda_d(x)\right) \mathrm{d}x \leq \liminf_{n \to \infty} \int_\omega F\left(x, \Lambda_{d_n}(x)\right) \mathrm{d}x,$$

and the claim immediately follows by taking the supremum over k of the left-hand side term and by the monotone convergence theorem.

We now come to the proof of Theorem 4.3. The proof is essentially divided in two steps: first, we approximate a given $d \in \mathcal{D}$ with distances induced by a sequence of Borel measurable and isotropic Riemannian metrics, then we approximate each distance of the sequence by means of distances in $\mathcal{I}$.

Proposition 4.6. *Let $d \in \mathcal{D}$. Then there exists a sequence of Borel measurable isotropic metrics $a_n : \Omega \to [\alpha, \beta]$ such that*

(i) $d_{a_n} \xrightarrow{\mathcal{D}} d$;

(ii) $a_n(x) = \Lambda_d(x)$ *for a.e.* $x \in \Omega$.

Proof : By Lemma 3.7, it is sufficient to define the functions a_n in such a way that the generated distances d_{a_n} converges pointwise to d on a dense subset of $\Omega \times \Omega$. Let us start then by setting $S := \mathbb{Q}^N \cap \Omega$. Obviously $S \times S$ is dense in $\Omega \times \Omega$ and countable, so we write $S \times S := \{(x_i, y_i) \,|\, i \in \mathbb{N}\}$. For each (x_i, y_i) we take a d-minimizing sequence $(\gamma_n^i)_{n \in \mathbb{N}} \subset \mathcal{L}_{x_i, y_i}$, i.e. such that

$$d(x_i, y_i) \leq \mathrm{L}_d(\gamma_n^i) \leq d(x_i, y_i) + \frac{1}{n}. \tag{20}$$

By Lemma 3.9, the curves γ_n^i can be chosen in such a way to satisfy conditions (i) and (ii) of the mentioned lemma (this assumption is not really needed here, but will be important in the proof of Theorem 4.3). By condition (ii), each non-empty set $\gamma_n^i(I) \cap \gamma_n^j(I)$ is a disjoint finite union of closed arcs. Let us denote by T_n the finite set given by the extreme points of such arcs for every $1 \leq i \leq j \leq n$ and set $N_n := \cup_{i \leq n} \gamma_n^i(I)$. Let Σ_n be a Borel $\mathcal{H}^1$-negligible subset of N_n which contains the points where the 1-rectifiable set N_n is not differentiable (this is possible by the regularity of the measure $\mathcal{H}^1$ and by the differentiability property of rectifiable sets [10, Theorem 1.6, Theorems 3.8 and 3.14]). Then we define the function $a_n : \Omega \to [\alpha, \beta]$ by

$$a_n(x) := \begin{cases} \Lambda_d(x) & \text{if } x \in \Omega \setminus N_n \\ \alpha & \text{if } x \in \Sigma_n \cup T_n \\ \varphi_d(x, \xi_x) & \text{if } x \in N_n \setminus (\Sigma_n \cup T_n) \end{cases} \tag{21}$$

where ξ_x is the unitary tangent to N_n at the point x. It is not difficult to prove that a_n is Borel-measurable. Moreover it is clear that a_n satisfies point (ii) of the Proposition.

We remark that, by [8, Corollary 2.7], we have that $\varphi_d(x, \xi_x) = \varphi_d(x, -\xi_x)$ for $\mathcal{H}^1$-a.e. $x \in N_n$. By possibly enlarging the set Σ_n we may suppose that this holds everywhere on $N_n \setminus \Sigma_n$. Moreover, if $x = \gamma_i^n(t)$ and γ_i^n is differentiable in t, we have that $\dot{\gamma}_i^n(t)$ is parallel to ξ_x and therefore $\varphi_d(\gamma_i^n(t), \dot{\gamma}_i^n(t)) = \varphi_d(\gamma_i^n(t), \xi_x)|\dot{\gamma}_i^n(t)| = a_n(\gamma_i^n(t))|\dot{\gamma}_i^n(t)|$.

Let d_{a_n} be the distances generated by such functions a_n. In order to prove point (i), we show that the distances d_{a_n} converge pointwise to d on $S \times S$. We claim that for every $i \leq n$ we have

$$d(x_i, y_i) \leq d_{a_n}(x_i, y_i) \leq d(x_i, y_i) + \frac{1}{n}.$$

Let us fix an $i \leq n$ and let us prove the second inequality. By the above remark and by (20) we have

$$d_{a_n}(x_i, y_i) \leq \int_0^1 a_n(\gamma_n^i)|\dot{\gamma}_n^i| \, \mathrm{d}t = \int_0^1 \varphi_d(\gamma_n^i, \dot{\gamma}_n^i) \, \mathrm{d}t \leq d(x_i, y_i) + \frac{1}{n}.$$

To prove the first inequality, choose a curve $\sigma \in \mathcal{L}_{x_i,y_i}$ and for every $i \leq n$ set $I_i := \{t \in I \,|\, \sigma(t) \in \gamma_n^i(I)\}$ and $I_0 := I \setminus \cup_{i \leq n} I_i$. We remark that the vector $\dot{\sigma}(t)$ is parallel to $\xi_{\sigma(t)}$ a.e. on each I_i and so $a_n(\sigma)|\dot{\sigma}| = \varphi_d(\sigma, \dot{\sigma})$ a.e. on I_i. Therefore we have

$$\begin{aligned} \mathrm{L}_{a_n}(\sigma) &= \int_0^1 a_n(\sigma)|\dot{\sigma}| \, \mathrm{d}t = \sum_{i=1}^n \int_{I_i} a_n(\sigma)|\dot{\sigma}| \, \mathrm{d}t + \int_{I_0} a_n(\sigma)|\dot{\sigma}| \, \mathrm{d}t \\ &\geq \sum_{i=1}^n \int_{I_i} \varphi_d(\sigma, \dot{\sigma}) \, \mathrm{d}t + \int_{I_0} \varphi_d(\sigma, \dot{\sigma}) \, \mathrm{d}t \geq d(x_i, y_i), \end{aligned}$$

where we have used the fact that $a_n(\sigma)|\dot{\sigma}| \geq \varphi_d(\sigma, \dot{\sigma})$ on I_0. By passing to the infimum over all possible curves $\sigma \in \mathcal{L}_{x_i,y_i}$ we get the claim. □

Proof of Theorem 4.3. The proof is organized in two steps.

Step 1. We first remark that the closure of $\mathcal{I}$ contains the family of distances generated by lower semicontinuous isotropic Riemannian metrics. In fact, let $b : \Omega \to [\alpha, \beta]$ be a lower semicontinuous metric. It is well known that $b(x) = \sup_{n \in \mathbb{N}} \tilde{a}_n(x)$ for suitable continuous functions $\tilde{a}_n$ (and we may as well suppose that $\alpha \leq \tilde{a}_n \leq \beta$ by possibly replacing the function $\tilde{a}_n$ with $\tilde{a}_n \vee \alpha$). Setting $a_n(x) := \sup_{i \leq n} \tilde{a}_i(x)$, we have that $d_{a_n} \xrightarrow{\mathcal{D}} d_b$ by Lemma 3.8. Moreover, by Proposition 3.5 we have that $\Lambda_{d_b}(x) = b(x)$ and $\Lambda_{d_{a_n}}(x) = a_n(x)$ almost everywhere on Ω and therefore, by the monotone convergence, we get that

$$\limsup_n \int_\Omega F(x, \Lambda_{d_{a_n}}(x)) \, \mathrm{d}x = \int_\Omega F(x, \Lambda_b(x)) \, \mathrm{d}x.$$

To prove the theorem, it is then sufficient to find a sequence of lower semicontinuous metrics $b_n : \Omega \to [\alpha, \beta]$ such that the generated distances d_{b_n} satisfy the claim of the theorem. Indeed, by combining the idea just described with a diagonal argument, the conclusion would follow at once.

Step 2. To get the desired approximation of the distance $d \in \mathcal{D}$ via lower semicontinuous isotropic metrics, it is enough to prove that, for every fixed $n \in \mathbb{N}$ there exists a sequence of lower semicontinuous isotropic metrics $b_k : \Omega \to [\alpha, \beta]$ such that

(i) $d(x_i, y_i) \leq \limsup_{k \to +\infty} d_{b_k}(x_i, y_i) \leq d(x_i, y_i) + \frac{1}{n}$ for every $i \leq n$;

(ii) $\limsup_{k \to +\infty} \int_\Omega F(x, b_k(x)) \, \mathrm{d}x \leq \int_\Omega F(x, a_n(x)) \, \mathrm{d}x$

where a_n are the Borel isotropic metrics built in the proof of Proposition 4.6.

In fact the desired sequence of lower semicontinuous metrics is then obtained via a diagonal argument and taking into account that $a_n(x) = \Lambda_d(x)$ almost everywhere on Ω by Proposition 4.6.

Keeping the notation used in the proof of Proposition 4.6, we observe that the set $N_n \setminus T_n$ is a finite, disjoint union of open arcs. Therefore, by applying Lemma 3.10 to each arc, we can find a sequence of continuous functions $\sigma_k : N_n \setminus T_n \to [\alpha, \beta]$ which converge to a_n $\mathcal{H}^1$-a.e. on $N_n \setminus T_n$. Let us set $A_k := \{x \in \Omega \,|\, \mathrm{dist}(x, N_n) < 1/k\}$. Let $(\Omega_k)_{k\in\mathbb{N}}$ be a sequence of bounded open sets well contained in Ω such that $\overline{\Omega}_k \subset \Omega_{k+1}$ and $\Omega = \bigcup_{k\in\mathbb{N}} \Omega_k$. By Lusin's theorem we may find a sequence of closed set $K_k \subset \Omega_k \setminus A_k$ such that $a\,|_{K_k}$ is continuous and $\mathcal{L}^n((\Omega_k \setminus A_k) \setminus K_k) < 1/k$. Then we define $b_k : \Omega \to [\alpha, \beta]$ by

$$b_k(x) := \begin{cases} \sigma_k(x) & \text{if } x \in N_n \setminus T_n \\ \alpha & \text{if } x \in T_n \\ a_n(x) & \text{if } x \in K_k \\ \beta & \text{elsewhere.} \end{cases} \tag{22}$$

Notice that b_k is lower semicontinuous. Moreover we have

$$\limsup_{k\to+\infty} \int_\Omega F(x, b_k(x))\,\mathrm{d}x = \limsup_{k\to+\infty} \left(\int_{K_k} F(x, a_n(x))\,\mathrm{d}x + \int_{\Omega\setminus K_k} F(x,\beta)\,\mathrm{d}x \right). \tag{23}$$

Recalling that $F(x, \beta)$ is summable over Ω (condition (ii) of (6)), we have that the second integral in the right-hand side of (23) goes to zero. In fact

$$\int_{\Omega\setminus K_k} F(x,\beta)\,\mathrm{d}x = \int_{\Omega\setminus\Omega_k} F(x,\beta)\,\mathrm{d}x + \int_{\Omega_k\setminus K_k} F(x,\beta)\,\mathrm{d}x, \tag{24}$$

and the first and second term of the right-hand side of (24) go to zero, respectively by the dominated convergence theorem and the absolute continuity of the integral. Therefore

$$\limsup_{k\to+\infty} \int_\Omega F(x, b_k(x))\,\mathrm{d}x \le \int_\Omega F(x, a_n(x))\,\mathrm{d}x,$$

so point (ii) of the claim is satisfied.
Let us show now that (i) holds. We start by proving the second inequality. For $i \le n$ we have by definition

$$d_{b_k}(x_i, y_i) \le \mathrm{L}_{b_k}(\gamma_i^n) = \int_0^1 \sigma_k(\gamma_i^n)|\dot\gamma_i^n|\,\mathrm{d}t,$$

therefore by the dominated convergence theorem we get

$$\begin{aligned} \limsup_{k\to+\infty} d_{b_k}(x_i, y_i) &\le \limsup_{k\to+\infty} \int_0^1 \sigma_k(\gamma_i^n)|\dot\gamma_i^n|\,\mathrm{d}t = \int_0^1 a_n(\gamma_i^n)|\dot\gamma_i^n|\,\mathrm{d}t \\ &= \int_0^1 \varphi_d(\gamma_i^n, \dot\gamma_i^n)\,\mathrm{d}t \le d(x_i, y_i) + \frac{1}{n}. \end{aligned} \tag{25}$$

To prove the first inequality let us take for every $k \in \mathbb{N}$ a curve $\gamma_k \in \mathcal{L}_{x_i, y_i}$ such that

$$d_k(x_i, y_i) \le \mathrm{L}_{b_k}(\gamma_k) \le d_k(x_i, y_i) + \frac{1}{k}. \tag{26}$$

Once again, we remark that, by Lemma 3.9, it is not restrictive to suppose that such curves are injective. Since $\alpha \int_I |\dot\gamma_k|\,\mathrm{d}t \le \mathrm{L}_{b_k}(\gamma_k)$, by (26) and (25) we get that $\limsup_k \int_I |\dot\gamma_k|\,\mathrm{d}t < +\infty$. Let us choose an $\varepsilon > 0$. By applying Lemma 3.10 to each open arc of $N_n \setminus T_n$, we can find a Borel set $B_\varepsilon \subset N_n \setminus T_n$ and an infinitesimal sequence of positive numbers $(\delta_k)_{k\in\mathbb{N}}$ such that $\mathcal{H}^1(N_n \setminus B_\varepsilon) < \varepsilon$ and $|\sigma_k(x) - a_n(x)| < \delta_k$ for every $x \in B_\varepsilon$. Let us set $I_k := \{t \in I | \gamma_k(t) \in N_n \setminus B_\varepsilon\}$. Then $b_k(\gamma_k) \ge a_n(\gamma_k) - \delta_k$ a.e. on $I \setminus I_k$. Let us write

$$L_{b_k}(\gamma_k) = \int_{I_k} b_k(\gamma_k)|\dot\gamma_k|\mathrm{d}t + \int_{I\setminus I_k} b_k(\gamma_k)|\dot\gamma_k|\mathrm{d}t.$$

We remark that, as $\gamma_k(I_k) \subset N_n \setminus B_\varepsilon$ for every $k \in \mathbb{N}$, by the Area-formula we have

$$\int_{I_k} |\dot\gamma_k|\,\mathrm{d}t = \mathcal{H}^1(\gamma_k(I_k)) \le \mathcal{H}^1(N_n \setminus B_\varepsilon) < \varepsilon.$$

Taking into account this remark we get

$$\begin{aligned}\int_{I_k} b_k(\gamma_k)|\dot\gamma_k|\mathrm{d}t &= \int_{I_k} a_n(\gamma_k)|\dot\gamma_k|\mathrm{d}t + \int_{I_k} (b_k(\gamma_k) - a_n(\gamma_k))|\dot\gamma_k|\mathrm{d}t \\ &\ge \int_{I_k} a_n(\gamma_k)|\dot\gamma_k|\mathrm{d}t - (\beta-\alpha)\varepsilon.\end{aligned}$$

Then we have

$$\begin{aligned}\mathrm{L}_{b_k}(\gamma_k) &\ge \int_0^1 a_n(\gamma_k)|\dot\gamma_k|\,\mathrm{d}t - \delta_k \int_{I\setminus I_k} |\dot\gamma_k|\,\mathrm{d}t - (\beta-\alpha)\varepsilon \\ &\ge d_{a_n}(x_i,y_i) - \delta_k \int_0^1 |\dot\gamma_k|\,\mathrm{d}t - (\beta-\alpha)\varepsilon\end{aligned}$$

and therefore, as $\delta_k \int_0^1 |\dot\gamma_k|\,\mathrm{d}t$ goes to zero, we obtain

$$\limsup_{k\to+\infty} d_{b_k}(x_i,y_i) \ge \limsup_{k\to+\infty} \mathrm{L}_{b_k}(\gamma_k) \ge d_{a_n}(x_i,y_i) - (\beta-\alpha)\varepsilon.$$

The claim then follows since ε was arbitrary. □

Remark 4.7. It should be noticed that the proof of Theorem 4.3 holds under very general assumptions on the function F, namely it is sufficient to take an F which is Borel measurable and satisfies assumption (ii) of (6), and such that the function $F(x,\cdot)$ is non-decreasing for $\mathcal{L}^N$-a.e. $x \in \Omega$. This consideration, together with Remark 4.5, enables us to conclude that our relaxation result, namely Theorem 4.1, holds under the following milder conditions on $F : \Omega \times [\alpha,\beta] \to \mathbb{R}$:

(i) there exist a sequence of continuous functions $F_k : \Omega \times [\alpha,\beta] \to \mathbb{R}$ satisfying conditions (6) and such that $F(x,\xi) = \sup_{k\in\mathbb{N}} F_k(x,\xi)$ for $\mathcal{L}^N$-a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^N$;

(ii) $\int_\Omega F(x,\beta)\,\mathrm{d}x < +\infty$.

With a slight modification of the argument used in the proof of Proposition 4.6 we can prove the following result.

Proposition 4.8. *Let $d \in \mathcal{D}$. Then there exists a Borel function $a : \overline{\Omega} \to [\alpha, \beta]$ such that, for every $(x, y) \in \Omega \times \Omega$,*

$$d(x,y) = \inf\left\{ \int_0^1 a(\gamma(t))|\dot{\gamma}(t)|\,\mathrm{d}t \;:\; \gamma \in \mathrm{Lip}([0,1];\overline{\Omega}),\ \gamma(0) = x,\ \gamma(1) = y \right\}.$$

In particular, if $\Omega := \mathbb{R}^N$, for every $d \in \mathcal{D}$ there exists a Borel measurable, isotropic Riemannian metric $a : \mathbb{R}^N \to [\alpha, \beta]$ such that $d = d_a$ according to definition (2).

Proof : Let us first remark that one can think the distance $d \in \mathcal{D}$ to be defined on $\overline{\Omega} \times \overline{\Omega}$ by extending it continuously up to the boundary. Therefore the d-length of every path $\gamma : I \to \overline{\Omega}$ is defined, according to definition (9). Let us define the *metric derivative* of the path γ at the point $t \in I$ as

$$\mathrm{md}_d(\gamma)(t) := \lim_{h \to 0} \frac{d(\gamma(t+h), \gamma(t))}{h}. \tag{27}$$

It is well known (see [2] for instance) that the limit in (27) exists for $\mathcal{L}^1$-a.e. $t \in I$ and that

$$\mathrm{L}_d(\gamma) = \int_0^1 \mathrm{md}_d(\gamma)(t)\,\mathrm{d}t.$$

Notice also that, if $\gamma(t) \in \Omega$, then $\mathrm{md}_d(\gamma)(t) = \varphi_d(\gamma(t), \dot{\gamma}(t))$, as one can easily show comparing the definitions of φ_d and md_d and recalling that locally $\alpha|x-y| \le d(x,y) \le \beta|x-y|$. Moreover, we observe that a Borel function $a : \overline{\Omega} \to [\alpha, \beta]$ induces a distance δ_a on $\overline{\Omega}$ through the formula

$$\delta_a(x,y) := \inf\left\{ \int_0^1 a(\gamma(t))|\dot{\gamma}(t)|\,\mathrm{d}t \;:\; \gamma \in \mathrm{Lip}([0,1];\overline{\Omega}),\ \gamma(0) = x,\ \gamma(1) = y \right\} \tag{28}$$

for every $(x, y) \in \overline{\Omega} \times \overline{\Omega}$.

Comparing the definition of δ_a with the one of d_a given in (2), we see that the main difference relies upon the fact that the curves on which we minimize the length L_a are now allowed to lye in the closure of Ω, therefore δ_a depends also from the values assumed by a on the boundary of Ω. In particular, we remark that in general $\delta_a(x,y) \le d_a(x,y)$ for $(x,y) \in \Omega \times \Omega$, and this inequality may be strict due to the fact that a is not continuous. For instance, take $\Omega := (-1,1) \times (-1,1)$ and $a(x) := \chi_{\overline{\Omega}}(x) + \chi_{\Omega}(x)$. One can easily see that points near the boundary of Ω are closer with respect to δ_a since also the boundary of Ω can be used to connect points in definition (28).

Let us now set $S := \mathbb{Q}^N \cap \Omega$ and write $S \times S = \{(x_i, y_i) \,|\, i \in \mathbb{N}\}$. By the lower semicontinuity of L_d, we have that for every $i \in \mathbb{N}$ there exists a curve $\gamma_i : I \to \overline{\Omega}$ such that $\mathrm{L}_d(\gamma_i) = d(x_i, y_i)$ (just take for γ_i an accumulation point of a d-minimizing sequence of curves in $\overline{\Omega}$ which connect x and y). Let $N_n := \cup_{i \le n} \gamma_i(I)$ and Σ_n be an $\mathcal{H}^1$-negligible Borel set which contains the non-differentiability points of N_n. Then define $a_n : \overline{\Omega} \to [\alpha, \beta]$ by

$$a_n(x) := \begin{cases} \alpha & \text{if } x \in \Sigma_n \\ \frac{\mathrm{md}_d(\gamma_i)(t)}{|\dot{\gamma}_i(t)|} & \text{if } x = \gamma_i(t) \in N_n \setminus \Sigma_n \text{ for some } i \le n \text{ and some } t \in I \\ \beta & \text{elsewhere.} \end{cases} \tag{29}$$

It is easy to show that a_n is Borel measurable. Moreover, arguing as in the proof of Proposition 4.6, one can show that $\delta_{a_n}(x_i, y_i) = d(x_i, y_i)$ for every $i \le n$. Notice that $N_n \subset N_{n+1}$

and, up to replacing Σ_{n+1} with $\Sigma_{n+1} \cup \Sigma_n$, we can always suppose that $\Sigma_n \subset \Sigma_{n+1}$. Therefore $(a_n)_{n\in\mathbb{N}}$ is a decreasing sequence of metrics. Let $a(x) := \inf_{n\in\mathbb{N}} a_n(x)$. Then, arguing as in Lemma 3.8, we get that

$$\delta_a(x_i, y_i) = \lim_{n\to+\infty} \delta_{a_n}(x_i, y_i) = d(x_i, y_i)$$

for every $i \in \mathbb{N}$. This means that $\delta_a = d$ on a dense subset of $\Omega \times \Omega$ and hence δ_a coincides with d by continuity, which is the claim. □

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